Valued Difference Fields

Gönenç Onay (joint with S.Durhan)

Mimar Sinan Güzel Sanatlar Üniversitesi

Model Theory In Wroclaw 2012

Valued difference fields

- Let (K, v) be a valued field, with k its residue field and $\mathcal{O}_v$ its valuation ring.
- If $\sigma \in Aut(K)$ with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$ then σ induces automorphisms:
  σ_v on $v(K)$ and $\overline{\sigma}$ on k; σ_v strickly increasing:
  $\gamma < \delta \Rightarrow \sigma(\gamma) < \sigma(\delta)$
- $(k, \bar{\sigma})$ is a difference field

In this case we say that (K, v, σ) is a **valued difference field.**

Remark

Any perfect valued field of equal characteristic $(p > 0)$ is a valued difference field.

Valued difference fields

- Let (K, v) be a valued field, with k its residue field and $\mathcal{O}_v$ its valuation ring.
- If $\sigma \in Aut(K)$ with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$ then σ induces automorphisms:
  σ_v on $v(K)$ and $\overline{\sigma}$ on k; σ_v strickly increasing:
  $\gamma < \delta \Rightarrow \sigma(\gamma) < \sigma(\delta)$
- $(k, \bar{\sigma})$ is a difference field

In this case we say that (K, v, σ) is a **valued difference field.**

Remark

Any perfect valued field of equal characteristic $(p > 0)$ is a valued difference field.

Valued difference fields

- Let (K, v) be a valued field, with k its residue field and $\mathcal{O}_v$ its valuation ring.
- If $\sigma \in Aut(K)$ with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$ then σ induces automorphisms:
  σ_v on $v(K)$ and $\overline{\sigma}$ on k; σ_v strickly increasing:
  $\gamma < \delta \Rightarrow \sigma(\gamma) < \sigma(\delta)$
- $(k, \bar{\sigma})$ is a difference field

In this case we say that (K, v, σ) is a **valued difference field.**

Remark

Any perfect valued field of equal characteristic $(p > 0)$ is a valued difference field.

Valued difference fields

- Let (K, v) be a valued field, with k its residue field and $\mathcal{O}_v$ its valuation ring.
- If $\sigma \in Aut(K)$ with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$ then σ induces automorphisms:
  σ_v on $v(K)$ and $\overline{\sigma}$ on k; σ_v strickly increasing:
  $\gamma < \delta \Rightarrow \sigma(\gamma) < \sigma(\delta)$
- $(k, \bar{\sigma})$ is a difference field

In this case we say that (K, v, σ) is a **valued difference field.**

Remark

Any perfect valued field of equal characteristic ($p > 0$) is a valued difference field.

σ-polynomials

σ-polynomials: A finite sum of σ-monomials which are of the form

$$ax^{i_0}(\sigma(x))^{i_1}\dots(\sigma^n(x))^{i_n},$$

where a is said to be the **coefficient** of M and the $n+1$-tuple $(i_0, i_1, \dots, i_n)$ be the **index** of M, denoted by $ind(M)$. We consider $n+1$ tuples of integers under the partial ordering induced by $\mathbb{N}$.

linear σ-polynomials:

$$a_0x + a_1\sigma(x) + \dots + a_n\sigma^n(x).$$

Remark

σ-linear polynomials (:polynomial functions) has a (non-commutative) ring structure under addition and composition. K is a module over this ring.

σ-polynomials

σ-polynomials: A finite sum of σ-monomials which are of the form

$$ax^{i_0}(\sigma(x))^{i_1}\ldots(\sigma^n(x))^{i_n},$$

where a is said to be the **coefficient** of M and the $n+1$-tuple $(i_0, i_1, \ldots, i_n)$ be the **index** of M, denoted by $ind(M)$. We consider $n+1$ tuples of integers under the partial ordering induced by $\mathbb{N}$.

linear σ-polynomials:

$$a_0x + a_1\sigma(x) + \cdots + a_n\sigma^n(x).$$

Remark

σ-linear polynomials (:polynomial functions) has a (non-commutative) ring structure under addition and composition. K is a module over this ring.

σ-polynomials

σ-polynomials: A finite sum of σ-monomials which are of the form

$$ax^{i_0}(\sigma(x))^{i_1}\dots(\sigma^n(x))^{i_n},$$

where a is said to be the **coefficient** of M and the $n+1$-tuple $(i_0, i_1, \dots, i_n)$ be the **index** of M, denoted by $ind(M)$. We consider $n+1$ tuples of integers under the partial ordering induced by $\mathbb{N}$.

linear σ-polynomials:

$$a_0x + a_1\sigma(x) + \dots + a_n\sigma^n(x).$$

Remark

σ-linear polynomials (:polynomial functions) has a (non-commutative) ring structure under addition and composition. K is a module over this ring.

σ-polynomials

σ-polynomials: A finite sum of σ-monomials which are of the form

$$ax^{i_0}(\sigma(x))^{i_1}\ldots(\sigma^n(x))^{i_n},$$

where a is said to be the **coefficient** of M and the $n+1$-tuple $(i_0, i_1, \ldots, i_n)$ be the **index** of M, denoted by $ind(M)$. We consider $n+1$ tuples of integers under the partial ordering induced by $\mathbb{N}$.

linear σ-polynomials:

$$a_0x + a_1\sigma(x) + \cdots + a_n\sigma^n(x).$$

Remark

σ-linear polynomials (:polynomial functions) has a (non-commutative) ring structure under addition and composition. K is a module over this ring.

$\mathbb{Z}[\sigma_v]$-module $v(K)$

- For $\gamma \in v(K)$, $x \in K$ with $v(x) = \gamma$ and a σ monomial M, we set

  $\gamma \cdot M := v(a_j x^{i_0}(\sigma(x))^{i_1} \ldots (\sigma^n(x))^{i_n}) = v(a_j) + \sum_{j=0}^{n} i_j \sigma_v^j(\gamma)$,

The action of the $\{\cdot M \mid M \text{ a } \sigma\text{-monomial with coefficient } 1\}$ induces a $\mathbb{Z}[\sigma_v]$-module structure on $v(K^\times)$.

$\mathbb{Z}[\sigma_v]$-module $v(K)$

- For $\gamma \in v(K)$, $x \in K$ with $v(x) = \gamma$ and a σ monomial M, we set
  $\gamma \cdot M := v(a_j x^{i_0}(\sigma(x))^{i_1} \dots (\sigma^n(x))^{i_n}) = v(a_j) + \sum_{j=0}^{n} i_j \sigma_v^j(\gamma)$,

The action of the $\{\cdot M \mid M \text{ a } \sigma\text{-monomial with coefficient } 1\}$ induces a $\mathbb{Z}[\sigma_v]$-module structure on $v(K^\times)$.

Ax-Kochen and Ershov Principles

We want to have that: Given two valued difference fields (K, v, σ) and (K', v', σ') such that

- $(k, \bar{\sigma}) \equiv (k', \bar{\sigma'})$ as difference fields and
- $v(K^{\times}) \equiv v(K'^{\times})$ as $\mathbb{Z}[\sigma]$-modules

then $(K, v, \sigma) \equiv (K', v', \sigma')$ as valued difference fields.

Ax-Kochen and Ershov Principles

We want to have that: Given two valued difference fields (K, v, σ) and (K', v', σ') such that

- $(k, \bar{\sigma}) \equiv (k', \bar{\sigma'})$ as difference fields and
- $v(K^\times) \equiv v(K'^\times)$ as $\mathbb{Z}[\sigma]$-modules

then $(K, v, \sigma) \equiv (K', v', \sigma')$ as valued difference fields.

Ax-Kochen and Ershov Principles

We want to have that: Given two valued difference fields (K, v, σ) and (K', v', σ') such that

- $(k, \bar{\sigma}) \equiv (k', \bar{\sigma'})$ as difference fields and
- $v(K^\times) \equiv v(K'^\times)$ as $\mathbb{Z}[\sigma]$-modules

then $(K, v, \sigma) \equiv (K', v', \sigma')$ as valued difference fields.

Ax-Kochen and Ershov Principles

We want to have that: Given two valued difference fields (K, v, σ) and (K', v', σ') such that

- $(k, \bar{\sigma}) \equiv (k', \bar{\sigma'})$ as difference fields and
- $v(K^{\times}) \equiv v(K'^{\times})$ as $\mathbb{Z}[\sigma]$-modules

then $(K, v, \sigma) \equiv (K', v', \sigma')$ as valued difference fields.

Compatible couples of functions

Definition

Let (K, v) be a valued field (or any expansion of a valued abelian group) and $A \subseteq K$. A couple of functions (f, f_v) where $f : K \to K$ and $f_v : v(K) \to v(K)$ is said to be A-**compatible** (or compatible on A) if $v(f(a)) = f_v(v(a))$ for all $a \in A$.

Example

- a (σ)-monomial couple: $(M, \cdot M)$ is everywhere compatible.
- (σ, σ_v) is everywhere compatible.

Compatible couples of functions

Definition

Let (K, v) be a valued field (or any expansion of a valued abelian group) and $A \subseteq K$. A couple of functions (f, f_v) where $f : K \to K$ and $f_v : v(K) \to v(K)$ is said to be A-**compatible** (or compatible on A) if $v(f(a)) = f_v(v(a))$ for all $a \in A$.

Example

- a (σ)-monomial couple: $(M, \cdot M)$ is everywhere compatible.
- (σ, σ_v) is everywhere compatible.

Compatible couples of functions

Definition

Let (K, v) be a valued field (or any expansion of a valued abelian group) and $A \subseteq K$. A couple of functions (f, f_v) where $f : K \to K$ and $f_v : v(K) \to v(K)$ is said to be *A*-**compatible** (or compatible on A) if $v(f(a)) = f_v(v(a))$ for all $a \in A$.

Example

- a (σ)-monomial couple: $(M, \cdot M)$ is everywhere compatible.
- (σ, σ_v) is everywhere compatible.

Compatible couples of functions

Definition

Let (K, v) be a valued field (or any expansion of a valued abelian group) and $A \subseteq K$. A couple of functions (f, f_v) where $f : K \to K$ and $f_v : v(K) \to v(K)$ is said to be A-**compatible** (or compatible on A) if $v(f(a)) = f_v(v(a))$ for all $a \in A$.

Example

- a (σ)-monomial couple: $(M, \cdot M)$ is everywhere compatible.
- (σ, σ_v) is everywhere compatible.

Polynomial couples $(P, \cdot P)$

- For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set $\gamma \cdot P := \min_j\{\gamma \cdot M_j\}$
- **!:** $(P, \cdot P)$ is in general not a compatible couple: if x a non-zero root of P, $v(P(x)) = \infty > v(x) \cdot P$.

We have the following general philosophy:

In nice valued structures, definable functions of one variable (polynomials, σ-polynomials, linear σ-polynomials, additives polynomials etc..) induces couples that we can easily describe where they are incompatible.
$\rightsquigarrow$ *Local Inversion Principles*

Polynomial couples $(P, \cdot P)$

- For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set $\gamma \cdot P := \min_j\{\gamma \cdot M_j\}$
- **!:** $(P, \cdot P)$ is in general not a compatible couple: if x a non-zero root of P, $v(P(x)) = \infty > v(x) \cdot P$.

We have the following general philosophy:

In nice valued structures, definable functions of one variable (polynomials, σ-polynomials, linear σ-polynomials, additives polynomials etc..) induces couples that we can easily describe where they are incompatible.
$\rightsquigarrow$ *Local Inversion Principles*

Polynomial couples $(P, \cdot P)$

- For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set $\gamma \cdot P := \min_j\{\gamma \cdot M_j\}$
- **!:** $(P, \cdot P)$ is in general not a compatible couple: if x a non-zero root of P, $v(P(x)) = \infty > v(x) \cdot P$.

We have the following general philosophy:

In nice valued structures, definable functions of one variable (polynomials, σ-polynomials, linear σ-polynomials, additives polynomials etc..) induces couples that we can easily describe where they are incompatible.
$\rightsquigarrow$ *Local Inversion Principles*

Polynomial couples $(P, \cdot P)$

- For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set $\gamma \cdot P := \min_j \{\gamma \cdot M_j\}$
- **!:** $(P, \cdot P)$ is in general not a compatible couple: if x a non-zero root of P, $v(P(x)) = \infty > v(x) \cdot P$.

We have the following general philosophy:

In nice valued structures, definable functions of one variable (polynomials, σ-polynomials, linear σ-polynomials, additives polynomials etc..) induces couples that we can easily describe where they are incompatible.

$\rightsquigarrow$ *Local Inversion Principles*

Polynomial couples $(P, \cdot P)$

- For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set $\gamma \cdot P := \min_j\{\gamma \cdot M_j\}$
- **!:** $(P, \cdot P)$ is in general not a compatible couple: if x a non-zero root of P, $v(P(x)) = \infty > v(x) \cdot P$.

We have the following general philosophy:

In nice valued structures, definable functions of one variable (polynomials, σ-polynomials, linear σ-polynomials, additives polynomials etc..) induces couples that we can easily describe where they are incompatible.
$\rightsquigarrow$ *Local Inversion Principles*

Regularity

An element $a \in K$ is said to be **regular** for a (σ-) polynomial P, if $v(P(a)) = v(a) \cdot P$ (in other words if $(P, \cdot P)$ is $\{a\}$-compatible). Otherwise we say that it is **irregular**.

Remark

A “regular non-zero root” does not make sense and 0 is always a regular root of any polynomial without constant term.

We will consider polynomials without constant term and equations of type $P(x) = b$ ($b \neq 0$) and say that “a is a regular solution” if $P(a) = b$ with a regular for P.

Regularity

An element $a \in K$ is said to be **regular** for a (σ-) polynomial P, if $v(P(a)) = v(a) \cdot P$ (in other words if $(P, \cdot P)$ is $\{a\}$-compatible). Otherwise we say that it is **irregular**.

Remark

A "regular non-zero root" does not make sense and 0 is always a regular root of any polynomial without constant term.

We will consider polynomials without constant term and equations of type $P(x) = b$ ($b \neq 0$) and say that "a is a regular solution" if $P(a) = b$ with a regular for P.

Finding regular elements

To have an (A-K,E) principle, one may ask:
Given a (σ-)polynomial P (at least for P in some *significant class*) and $a \in K$,

- Q1 is there an element x, regular for P, s.t. $v(P(x)) = v(a)$? (Picking the valuation of a; $\rightsquigarrow$ *weak*-continuity of P)
- Q2 is there an element x, regular for P, s.t. $v(P(x) - a) > v(a)$? (Picking the residue class of a)
- Q3 is there an element x, regular for P, s.t. $P(x) = a$? (Picking a)

Finding regular elements

To have an (A-K,E) principle, one may ask:
Given a (σ-)polynomial P (at least for P in some *significant class*) and $a \in K$,

- Q1 is there an element x, regular for P, s.t. $v(P(x)) = v(a)$? (Picking the valuation of a; $\rightsquigarrow$ *weak*-continuity of P)
- Q2 is there an element x, regular for P, s.t. $v(P(x) - a) > v(a)$? (Picking the residue class of a)
- Q3 is there an element x, regular for P, s.t. $P(x) = a$? (Picking a)

Finding regular elements

To have an (A-K,E) principle, one may ask:
Given a (σ-)polynomial P (at least for P in some *significant class*) and $a \in K$,

- Q1 is there an element x, regular for P, s.t. $v(P(x)) = v(a)$? (Picking the valuation of a; $\rightsquigarrow$ *weak*-continuity of P)
- Q2 is there an element x, regular for P, s.t. $v(P(x) - a) > v(a)$? (Picking the residue class of a)
- Q3 is there an element x, regular for P, s.t. $P(x) = a$? (Picking a)

Finding regular elements

To have an (A-K,E) principle, one may ask:
Given a (σ-)polynomial P (at least for P in some *significant class*) and $a \in K$,

- Q1 is there an element x, regular for P, s.t. $v(P(x)) = v(a)$? (Picking the valuation of a; $\rightsquigarrow$ *weak*-continuity of P)
- Q2 is there an element x, regular for P, s.t. $v(P(x) - a) > v(a)$? (Picking the residue class of a)
- Q3 is there an element x, regular for P, s.t. $P(x) = a$? (Picking a)

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive (:$\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive ($:\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive (:$\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive (:$\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive $(:\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N})$ then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive (:$\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

First problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

- $Jump(P)$ is finite $\Rightarrow$ P is "continuous" : for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.
- If $P \in K[X]$ or if P is any σ-polynomial with σ contractive ($:\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.
- if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.
- if σ_v is auto-increasing then σ-linear polynomials are continous.

The case of Kaplansky fields

A linear σ-polynomial is one of the form:
$a_n\sigma^n(x) + \cdots + a_1 x$.

Let (K, v) is of equal characteristic (p, p) $(p > 0)$. An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form:
$a_n p^n(x) + \cdots + a_1 x$

Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

The case of Kaplansky fields

A linear σ-polynomial is one of the form: $a_n\sigma^n(x) + \cdots + a_1 x$.

Let (K, v) is of equal characteristic (p, p) $(p > 0)$. An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form: $a_n p^n(x) + \cdots + a_1 x$

Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

The case of Kaplansky fields

A linear σ-polynomial is one of the form:
$a_n\sigma^n(x) + \cdots + a_1 x$.

Let (K, v) is of equal characteristic (p, p) $(p > 0)$. An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form:
$a_n p^n(x) + \cdots + a_1 x$

Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

The case of Kaplansky fields

A linear σ-polynomial is one of the form:
$a_n\sigma^n(x) + \cdots + a_1 x$.

Let (K, v) is of equal characteristic (p, p) $(p > 0)$. An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form:
$a_n p^n(x) + \cdots + a_1 x$

Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

The case of Kaplansky fields

A linear σ-polynomial is one of the form:
$a_n\sigma^n(x) + \cdots + a_1 x$.

Let (K, v) is of equal characteristic (p, p) $(p > 0)$. An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form:
$a_n p^n(x) + \cdots + a_1 x$

Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

Two very similar caracterization of algebraically maximal Kaplasky fields

Theorem (O.)

A Kaplansky field is algebraically maximal if and only if every equation of the form $P(x) = b$ ($b \neq 0$), where $P \in K[X]$ is additive, has a regular solution.

Theorem (Durhan)

A Kaplansky field is algebraically maximal if and only if it is p-henselian.

Two very similar caracterization of algebraically maximal Kaplasky fields

Theorem (O.)

A Kaplansky field is algebraically maximal if and only if every equation of the form $P(x) = b$ $(b \neq 0)$*, where* $P \in K[X]$ *is additive, has a regular solution.*

Theorem (Durhan)

A Kaplansky field is algebraically maximal if and only if it is p-henselian.

Algebraically maximal Kaplansky fields are *nice*:

- We have (A-K,E) principle for algebraically maximal Kaplansky fields.
- Theorem (O.): They are C-minimal as *valued modules* over the ring of its additive polynomials, whenever they have a divisible value group.

$\Rightarrow$ *significant class* $=$ {additive polynomials}

Algebraically maximal Kaplansky fields are *nice*:

- We have (A-K,E) principle for algebraically maximal Kaplansky fields.
- Theorem (O.): They are C-minimal as *valued modules* over the ring of its additive polynomials, whenever they have a divisible value group.

$\Rightarrow$ *significant class* = {additive polynomials}

Algebraically maximal Kaplansky fields are *nice*:

- We have (A-K,E) principle for algebraically maximal Kaplansky fields.
- Theorem (O.): They are C-minimal as *valued modules* over the ring of its additive polynomials, whenever they have a divisible value group.

$\Rightarrow$ *significant class* $=$ {additive polynomials}

Algebraically maximal Kaplansky fields are *nice*:

- We have (A-K,E) principle for algebraically maximal Kaplansky fields.
- Theorem (O.): They are C-minimal as *valued modules* over the ring of its additive polynomials, whenever they have a divisible value group.

$\Rightarrow$ *significant class* $=$ $\{$additive polynomials$\}$

General case

(*) We suppose $\overline{\sigma}^n \neq Id$ on k, for all $n \in \mathbb{N} \setminus \{0\}$.

Lemma (weak continuity)

Given a p.c. sequence $(a_\rho)_\Omega$ in K, $a \in K$, such that $(a_\rho)_\rho$ converges to a and a σ-polynomial P, we can find a p.c. sequence $(b_\lambda)_\lambda$ such that $(a_\rho)_\rho$ and $(b_\lambda)_\lambda$ have same limits , $(P(b_\rho))_\rho$ converges to $P(a)$.

Proof.

(Main trick) Using above assumption we can find $(b_\lambda)_{\lambda\in\Lambda}$ with Λ cofinal in Ω, such that $b_{\lambda+1} - b_\lambda$ is eventually regular for P. □

General case

(*) We suppose $\overline{\sigma}^n \neq Id$ on k, for all $n \in \mathbb{N} \setminus \{0\}$.

Lemma (weak continuity)

Given a p.c. sequence $(a_\rho)_\Omega$ in K, $a \in K$, such that $(a_\rho)_\rho$ converges to a and a σ-polynomial P, we can find a p.c. sequence $(b_\lambda)_\lambda$ such that $(a_\rho)_\rho$ and $(b_\lambda)_\lambda$ have same limits , $(P(b_\rho))_\rho$ converges to $P(a)$.

Proof.

(Main trick) Using above assumption we can find $(b_\lambda)_{\lambda\in\Lambda}$ with Λ cofinal in Ω, such that $b_{\lambda+1} - b_\lambda$ is eventually regular for P. □

Searching the significant class

From now on we consider the case of equal characteristic $(0, 0)$.

Definition

Given a σ-polynomial P we denote $Lin(P)$ the σ-linear part of P. Let $a \in K$, we say that (P, a) is in σ-hensel configuration if there exists $\gamma \in \Gamma$ such that

1. $v(P(a)) = \gamma \cdot Lin(P)$
2. $\gamma \cdot M < \gamma \cdot M'$ whenever M, M' are monomials of P such that $(0, \dots, 0) \neq ind(M) < ind(M')$.

Finding regular solutions

Lemma

Suppose that (K, v, σ) is σ-algebraically maximal and $(k, \bar{\sigma})$ is **linearly difference closed**, *that is: for every $\bar{\sigma}$-linear Q, and $c \in k$ the equation $Q(x) = c$ has solution in k.*
Conclusion: *For every σ-polynomial P and $b \in K^{\times}$ if for some $a \in K$ such that $v(P(a)) = b$, (P, a) is in σ-hensel configuration then there is a regular solution of the equation $P(x) = b$.*

Definition

(K, v, σ) is said to be σ-henselian if the conclusion of the above lemma holds.

Finding regular solutions

Lemma

Suppose that (K, v, σ) *is* σ*-algebraically maximal and* $(k, \bar{\sigma})$ *is* **linearly difference closed**, *that is: for every* $\bar{\sigma}$*-linear* Q, *and* $c \in k$ *the equation* $Q(x) = c$ *has solution in* k.

Conclusion: *For every* σ*-polynomial* P *and* $b \in K^{\times}$ *if for some* $a \in K$ *such that* $v(P(a)) = b$, (P, a) *is in* σ*-hensel configuration then there is a regular solution of the equation* $P(x) = b$.

Definition

(K, v, σ) is said to be σ-henselian if the conclusion of the above lemma holds.

Finding regular solutions

Lemma

Suppose that (K, v, σ) is σ-algebraically maximal and $(k, \bar{\sigma})$ is **linearly difference closed**, *that is: for every $\bar{\sigma}$-linear Q, and $c \in k$ the equation $Q(x) = c$ has solution in k.*

Conclusion: *For every σ-polynomial P and $b \in K^{\times}$ if for some $a \in K$ such that $v(P(a)) = b$, (P, a) is in σ-hensel configuration then there is a regular solution of the equation $P(x) = b$.*

Definition

(K, v, σ) is said to be σ-henselian if the conclusion of the above lemma holds.

Finding regular solutions

Lemma

Suppose that (K, v, σ) is σ-algebraically maximal and $(k, \bar{\sigma})$ is **linearly difference closed**, *that is: for every $\bar{\sigma}$-linear Q, and $c \in k$ the equation $Q(x) = c$ has solution in k.*
Conclusion: *For every σ-polynomial P and $b \in K^{\times}$ if for some $a \in K$ such that $v(P(a)) = b$, (P, a) is in σ-hensel configuration then there is a regular solution of the equation $P(x) = b$.*

Definition

(K, v, σ) is said to be σ-henselian if the conclusion of the above lemma holds.

Draft Results

- All algebraic, immediate, σ-algebraically maximal extensions of a valued difference field with a linearly difference closed residue field are isomorphic.
- (A-K,E) principle for holds for the class of σ-henselian valued difference fields of characteristic $(0,0)$ with linearly difference closed residue field.

Draft Results

- All algebraic, immediate, σ-algebraically maximal extensions of a valued difference field with a linearly difference closed residue field are isomorphic.
- (A-K,E) principle for holds for the class of σ-henselian valued difference fields of characteristic $(0, 0)$ with linearly difference closed residue field.