Geometria algebraiczna, Problem List 9

Let K be an algebraically closed field and $m, n \in \mathbb{N}_{>0}$.

(1) Let V be a finite subset of $\mathbb{P}^2$. Show that there is a line $L \subset \mathbb{P}^2$ such that $V \cap L = \emptyset$.

(2) Let us consider the natural action of $\mathrm{GL}_3(K)$ on K^3. Show that this action induces a transitive action of $\mathrm{GL}_3(K)$ on:
  (a) $\mathbb{P}^2$;
  (b) the set of two-dimensional K-linear subspaces of K^3;
  (c) the set of lines in $\mathbb{P}^2$.

(3) For $A \in \mathrm{GL}_3(K)$ and $F \in K[X, Y, Z]$, consider:

$$A \cdot F := F\left(A \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) \in K[X, Y, Z].$$

Show the following:
  (a) the above formula gives an action of $\mathrm{GL}_3(K)$ on $K[X, Y, Z]$;
  (b) for each $d \in \mathbb{N}$, the above action preserves the set of homogenous polynomials of degree d.

(4) Show that the action from Problem 3. gives an action of $\mathrm{GL}_3(K)$ on the set of algebraic subsets of $\mathbb{P}^2$ and that this action restricted to the set of lines in $\mathbb{P}^2$ coincides with the action from Problem 2(c).

(5) Let F, G be homogenous polynomials in $K[X, Y, Z]$, $x \in \mathbb{P}^2$, and $A \in \mathrm{GL}_3(K)$. Show that (using the previous problems to interpret the appropriate actions) we have:

$$I(x, F \cap G) = I(A \cdot x, (A \cdot F) \cap (A \cdot G)).$$

(6) Let

$$0 \to A_1 \to A_2 \to \ldots \to A_n \to 0$$

be an exact sequence of finite-dimensional vector spaces over K, that is for each $i \in \{1, \ldots, n\}$, we have:

$$\mathrm{im}(A_{i-1} \to A_i) = \ker(A_i \to A_{i+1}),$$

where $A_0 = 0 = A_{n+1}$.

Show the following "Inclusion–Exclusion Principle":

$$\sum_{i=1}^{n} (-1)^i \dim_K(A_i) = 0.$$

(7) For $k \in \mathbb{N}$, let R_k be the K-linear space consisting of homogenous polynomials of degree k in $K[X, Y, Z]$. Assume that $d \geqslant m + n$ and that we have an exact sequence of the form

$$0 \to R_{d-m-n} \to R_{d-n} \times R_{d-m} \to R_d \to E \to 0,$$

where E is a K-vector space. Show that:

$$\dim_K(E) = mn.$$

(8) For $F \in K[X_1, \ldots, X_n]$, let $F^* \in K[X_1, \ldots, X_{n+1}]$ be the homogenization of F with respect to X_{n+1}. Show that for all $F, G \in K[X_1, \ldots, X_n]$, we have:

$$X_{n+1}^t (F + G)^* = X_{n+1}^r F^* + X_{n+1}^s G^*,$$

where:

$$r = \deg(G), \quad s = \deg(F), \quad t = r + s - \deg(F + G).$$