Geometria algebraiczna, Problem List 8

Let K be an algebraically closed field and $n \in \mathbb{N}_{>0}$.

(1) Let X be a Noetherian topological space and
$$X = \bigcup_{i \in I} U_i$$
be an open cover of X. Show that:
$$\dim(X) = \sup_{i \in I} \dim(U_i).$$
(Therefore $\dim(\mathbb{P}^n) = n$.)

(2) Show that each two lines in $\mathbb{P}^2$ have a non-empty intersection.

(3) Let $H \in K[X_1, \ldots, X_n]$ and $d \in \mathbb{N}$. Show that the following are equivalent:
  (a) the polynomial H is homogenous of degree d;
  (b) for each $\lambda \in K$, we have the following equality of polynomials:
$$H(\lambda X_1, \ldots, \lambda X_n) = \lambda^d H.$$

(4) Let $F \in K[X_1, \ldots, X_n]$, $d \in \mathbb{N}$, and assume that:
  (a) $F = F_0 + \ldots + F_d$;
  (b) $F_d \neq 0$;
  (c) for each $i \leqslant d$, the polynomial F_i is homogenous of degree i or $F_i = 0$.

Show that:
$$\sum_{i=0}^{d} X_{n+1}^{d-i} F_i = X_{n+1}^{d} F\left(\frac{X_1}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}}\right).$$

(5) Find the points at infinity of the following projective plane curves and check whether these points are smooth:
  (a) $V(\alpha X + \beta Y + 1)$ for a fixed $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$;
  (b) $V(Y^2 - X^3)$;
  (c) $V(X^3 + Y^3 + 1)$;
  (d) the curves from Problem 5.8.