Geometria algebraiczna, Problem List 7

Let K be an algebraically closed field, R be a ring, and $m, n \in \mathbb{N}_{>0}$.

(1) Let $k_1, \ldots, k_n \in \mathbb{N}_{>0}$ and $F \in K[X] \setminus \{0\}$.
  (a) Show the following:
$$\dim_K(K[X]/(F)) = \deg(F).$$
  (b) Calculate the following:
$$\dim_K \left(K[X_1, \ldots, X_n]/(X_1^{k_1}, \ldots, X_n^{k_n})\right).$$
(2) Let $I, J \trianglelefteq R$ and $I \subseteq \sqrt{J}$. Show that if I is finitely generated, then there is $n \in \mathbb{N}$ such that $I^n \subseteq J$.
(3) Let $P \subseteq I \subseteq Q$ be ideals in a domain R such that P and Q are prime. Show that:
$$\frac{R_Q}{IR_Q} \cong \frac{(R/P)_{Q/P}}{I/P(R/P)_{Q/P}}.$$
(4) Let C be an affine curve, $v \in C$ be a smooth point, and $f \in K(C)$. Show that f is a local parameter for C at v if and only if f has a zero at v of order one.
(5) Let C be a plane curve and let $0 = (0,0) \in C$. Show that the tangent space $T_0(C)$ (understood here as $\pi_C^{-1}(0) \subseteq \mathbb{A}^2$, where $\pi_C : TC \to C$ is the projection morphism) is the union of tangent lines to C at 0.
(6) Calculate the following intersection numbers:
  (a) $I(0, (Y - X^2) \cap (Y^m + X^{2m}))$;
  (b) $I(0, (Y^4 + X^4 - X^2) \cap (Y^2 - X^3 + X^2))$;
  (c) $I(0, (Y^4X^3 + X^4Y^2 - X^2 + Y^7 + Y^2 + Y) \cap (Y^2 - Y^5X^3 + 1 + X^2))$;
  (d) $I(0, (XY^4 + X^4 - X^2 + X^8 + X) \cap (XY^2 - X^3 + X^2))$.
(7) Let S be a multiplicative subset of R.
  (a) Give the definition of the localization ring R_S (R need not be a domain!).
  (b) Describe the kernel of the following ring homomorphism:
$$\varphi : R \to R_S, \quad \varphi(r) = \frac{r}{1}.$$
  (c) Let P be a prime ideal in R and $e \in R \setminus P$ be such that $e^2 = e$ (an *idempotent* element). Let $\varphi : R \to R_P$ be as in Item (b) above. Show that φ restricted to eR induces an isomorphism of rings (with 1):
$$eR \cong R_P.$$
(8) Assume that $e_1, \ldots, e_m \in R$ satisfy the following:
  (a) $e_1^2 = e_1, \ldots, e_m^2 = e_m$;
  (b) $e_1 + \ldots + e_m = 1$;
  (c) for each $i \neq j$, we have $e_ie_j = 0$.
Show that the following function:
$$f : R \to e_1R \times \ldots \times e_mR, \quad f(r) = (re_1, \ldots, re_m)$$
is an isomorphism of rings (with 1).
(9) Assume that I, J are ideals in R such that $I + J = R$. Show that:
$$I^n + J^m = R.$$