Geometria algebraiczna, Problem List 3

Let R be a ring, K be an algebraically closed field, V, W be affine algebraic sets, and $m, n \in \mathbb{N}_{>0}$.

(1) Let R be UFD and $a, b \in R$ be such that $\mathrm{GCD}(a,b) = 1$. Show that:
$$R[X]/(aX+b) \cong_R R\left[b/a\right].$$
(2) Let $W \subset \mathbb{A}^n$ be finite of cardinality m.
  (a) Describe the ideal $I(W)$ as the intersection of maximal ideals of the ring $K[X_1, \ldots, X_n]$.
  (b) Using Chinese Remainder Theorem, show that:
$$K[W] \cong_K K^m.$$
  (c) Conclude that:
$$\mathrm{Fun}(W, K) = K[W].$$
  (d) Compare your proof of Item (c) above with your proof of Problem 1.19.

(3) Show that:
  (a) $V \times W$ is an affine algebraic set;
  (b) the set
$$\Delta_V := \{(v,v) \in V \times V \mid v \in V\} \text{ (the diagonal of } V\text{)}$$
  is Zariski closed;
  (c) if $A \subseteq V$ is Zariski dense in V and $F, G : V \to W$ are morphisms such that $F|_A = G|_A$, then $F = G$.

(4) A morphism $\Psi : V \to W$ is called a *monomorphism*, if for any affine algebraic set Z and any morphisms $\Phi_1, \Phi_2 : Z \to V$, we have the following:
$$\Phi_1 \neq \Phi_2 \quad \Rightarrow \quad \Psi \circ \Phi_1 \neq \Psi \circ \Phi_2.$$
A morphism $\Psi : V \to W$ is called an *epimorphism*, if for any affine algebraic set Z and any morphisms $\Phi_1, \Phi_2 : W \to Z$, we have the following:
$$\Phi_1 \neq \Phi_2 \quad \Rightarrow \quad \Phi_1 \circ \Psi \neq \Phi_2 \circ \Psi.$$
(These notions make sense in an arbitrary category replacing the category of affine algebraic sets.)
Describe monomorphisms and epimorphisms in the category of affine algebraic sets.

(5) Let $V = V(Y^2 - X^3) \subset \mathbb{A}^2$ and
$$\Psi : \mathbb{A}^1 \to V, \quad \Psi(x) = \left(x^2, x^3\right).$$
Show that Ψ is a bijective morphism, which is not an isomorphism.

(6) Let I be an ideal in a ring R. Show that the ring R/I is *reduced* (that is: R/I has no nilpotent elements) if and only if the ideal I is radical.