Geometria algebraiczna, Problem List 2

Let $K \subseteq L$ be an extension of fields, $A \subseteq L$, $a, b \in L$, and $n \in \mathbb{N}$.

(1) Show that there is a tower of fields $K \subseteq L_0 \subseteq L$ such that the extension $K \subseteq L_0$ is purely transcendental and the extension $L_0 \subseteq L$ is algebraic. (**Definition:** A field extension $K \subseteq L_0$ is *purely transcendental*, if there is $B \subseteq L_0$ such that B is algebraically independent over K and $L_0 = K(B)$.)

(2) Show that if $K \subseteq L$ is purely transcendental, then for each $x \in L \setminus K$, we have that x is transcendental over K.

(3) Find an example of a field extension $K \subseteq M$ such that:
   (a) For each $x \in M \setminus K$, x is transcendental over K, but the extension $K \subseteq M$ is not purely transcendental.
   (b) There is no tower of fields $K \subseteq L \subseteq M$ such that the extension $K \subseteq L$ is algebraic and the extension $L \subseteq M$ is purely transcendental.

(4) Assume that $a \notin A$. Show that $A \cup \{a\}$ is algebraically independent over K if and only if A is algebraically independent over K and a is transcendental over $K(A)$.

(5) Show that A is a transcendence basis of L over K if and only if A is algebraically independent over K and the extension $K(A) \subseteq L$ is algebraic. (**Definition:** A subset $A \subseteq L$ is a *transcendence basis* of L over K, if A is a maximal subset of L which is algebraically independent over K.)

(6) *Steinitz Exchange Principle*
Show that if a is algebraic over $K(A \cup \{b\})$ and a is transcendental over $K(A)$, then b is algebraic over $K(A \cup \{a\})$.

(7) Show that if B and B' are transcendental bases of L over K, then $|B| = |B'|$. (**Definition:** The cardinality of B above is called the *transcendence degree* of L over K and denoted $\mathrm{trdeg}_K L$.)

(8) Assume that M is a field, $\varphi : K \to M$ is an isomorphism and $K \subseteq K'$, $M \subseteq M'$ are algebraic closures. Show that φ extends to an isomorphism between K' and M'.

(9) Let $K \subseteq L'$ be a field extension such that:

$$\mathrm{trdeg}_K L = \mathrm{trdeg}_K L'.$$

Show that if L and L' are algebraically closed, then $L \cong_K L'$.

(10) Show that if L and L' are uncountable algebraically closed fields of the same characteristic and the same cardinality, then $L \cong L'$.

(11) Assume that $F \in K[X,Y]$ is irreducible and let $L := (K[X,Y]/(F))_0$. Show that $\mathrm{trdeg}_K L = 1$.

(12) The following result is called *Zariski's Lemma*:
"If $F \subseteq L$ is a field extension such that L is a finitely generated F-algebra, then $F \subseteq L$ is a finite field extension."
Show that Zariski's Lemma implies Weak Hilbert's Nullstellensatz.

(13) Let P be a prime ideal in a ring R and I, J be ideals in R such that $P = I \cap J$. Show that $P = I$ or $P = J$.

(14) Let R be a ring, $u \in R^*$, and $a \in R$. Show that the map

$$R \ni r \mapsto r + (uX + a) \in R[X]/(uX + a)$$

is a ring isomorphism.

(15) For which n, the polynomial $Y^2 - X^n \in K[X,Y]$ is irreducible?