Geometria algebraiczna, Problem List 11

Let K be an algebraically closed field and $n \in \mathbb{N}$.

(1) Let $x = [a : b : 1] \in \mathbb{P}^2$ and L be a line in $\mathbb{P}^2$ such that $x, [0 : 1 : 0] \in L$. Show that the affine line $L_* = L \cap \mathbb{A}^2$ is parallel to the y-axis.

(2) Assume that $K = \mathbb{C}$. Show that $\mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$ is compact in the Euclidean topology.

(3) Let $V \subseteq \mathbb{A}^n$ be an infinite Zariski closed subset. Show that $V^* \neq V$, where $V^* \subseteq \mathbb{P}^n$ is the homogenization of V.

(4) Let $A, B \in K$. We define:

$$E := V(Y^2 - X^3 - AX - B) \subseteq \mathbb{A}^2.$$

Show the following.

(a) The curve E has a unique point at infinity and this point is smooth.

(b) If $\mathrm{char}(K) \notin \{2, 3\}$, then E has at most one singular point.

(c) If $\mathrm{char}(K) \notin \{2, 3\}$, then E is smooth if and only if $4A^3 + 16B^2 \neq 0$.

(5) Let A, B, E be as in Problem (4) above. Assume that E^* is smooth and let us consider the elliptic curve (E^*, O), where O is the point at infinity of E. Let $(x_1, y_1), (x_2, y_2) \in E$ be two distinct points of E such that:

$$(x_3, y_3) := (x_1, y_1) \oplus (x_2, y_2) \in E.$$

Show that:

$$x_3 = \lambda^2 + a_1\lambda - a_2 - x_1 - x_2, \quad y_3 = -(\lambda + a_1)x_3 - \nu - a_3,$$

where the equation

$$Y = \lambda X + \nu$$

defines the line in $\mathbb{A}^2$ passing through points $(x_1, y_1), (x_2, y_2)$.

(6) Let $K = \mathbb{C}$ and

$$E := V(Y^2 - X^3 - 17) \subseteq \mathbb{A}^2.$$

Let (E^*, O) be an elliptic curve as in Problem (5) above. Check that:

$$P_1 := (-2, 3), P_2 := (-1, 4), P_3 = (2, 5) \in E$$

and compute the coordinates of the following points ($\oplus$ is the addition and $\ominus$ is the subtraction on the elliptic curve (E^*, O)):

(a) $P_1 \ominus P_3$;

(b) $P_2 \oplus P_2$;

(c) $P_2 \oplus P_3$.

(7) Let p be a prime number and

$$E_p := V\left(Y^2 - X^3 - 17\right) \subseteq \mathbb{A}^2\left(\mathbb{F}_p^{\mathrm{alg}}\right),$$

where $\mathbb{F}_p^{\mathrm{alg}}$ is the algebraic closure of $\mathbb{F}_p$. For which prime numbers p, the affine curve E_p is smooth?