Geometria algebraiczna, Problem List 10

Let K be an algebraically closed field and C be a smooth plane curve given by a homogenous polynomial $T \in K[X,Y,Z]$. Let $F, G \in K[X,Y,Z]$ be homogenous polynomials such that each of them has finitely many zeroes on C and all these zeroes are contained in $C_* := C \cap \mathbb{A}^2$. Recall that:

$$F_* := F|_{Z=1}.$$

(1) For $x \in C_*$, we say that the *Noether's condition holds for C, F, G* if:

$$F_* \in (T_*, G_*)K[X,Y]_{I(x)}.$$

We assume that Noether's condition holds for each $x \in C$ such that $G(x) = 0$. Show the following.

(a) There are $a, b \in K[X,Y]$ such that $F_* = aT_* + bG_*$.
Hint: use the following fact, which appeared in the proof of Proposition 2.56 and which says that if $I \triangleleft K[X,Y]$ and

$$V(I) = \{x_1, \ldots, x_n\}$$

is finite, then we have:

$$K[X,Y]/I \cong_K K[X,Y]_{I(x_1)}/IK[X,Y]_{I(x_1)} \times \ldots \times K[X,Y]_{I(x_n)}/IK[X,Y]_{I(x_n)}.$$

(b) There are $r \in \mathbb{N}$ and $a', b' \in K[X,Y,Z]$ such that

$$Z^r F = a'T + b'G.$$

Hint: use Problem 9.8

(c) There are $a'', b'' \in K[X,Y,Z]$ such that

$$F = a''T + b''G.$$

Hint: use the fact that a certain K-linear function from the proof of Bézout's theorem is one-to-one.

(d) There are $A, B \in K[X,Y,Z]$ homogenous and such that $F = AT + BG$ and moreover:

$$\deg(A) = \deg(F) - \deg(T), \ \deg(B) = \deg(F) - \deg(G).$$

Item (d) above is called the *$AF + BG$ Theorem* or *Max Noether's Fundamental Theorem* (we use "T" instead of "F" in the statement).

(2) Assume that:

$$I(x, C \cap F) \geqslant I(x, C \cap G).$$

Show that Noether's condition is satisfied by C, F, G and x.

(3) Assume that $A \in K[X,Y,Z]$ is homogenous and

$$\deg(A) = \deg(G) - \deg(T).$$

Show that:

$$T \cdot (G + AT) = T \cdot G.$$

(4) Show that if for each $x \in C$ we have

$$I(x, C \cap F) \geqslant I(x, C \cap G),$$

tthen there is a homogenous polynomial $H \in K[X,Y,Z]$ such that:

$$C \cdot F = C \cdot G + C \cdot H,$$

which is an equality in the group $\mathrm{Div}(C)$.