ALGEBRA 1R, Problem List 9

Let $n \in \mathbb{N}_{>0}$ and let G be a group.

(1) Show that:
$$Q_8' = \{I, -I\}.$$

(2) Show that (you *cannot* use that for $n \geqslant 5$, the group A_n is simple!):

   (a) For $n \geqslant 1$, we have:
$$(S_n)' = A_n.$$

   (b) For $n \geqslant 5$, we have
$$(A_n)' = A_n.$$

   (Hint: For $n \geqslant 3$, the group A_n is generated by the set of all 3-cycles.)

(3) Show that if $|G| = pq^2$, where p and q are prime numbers, then G is solvable.

(4) Show that if $|G| = 200$, then G is solvable.

(5) Show that if $|G| < 60$, then G is solvable.

(6) Find the largest number $n \in \mathbb{N}$, for which you can show that for any odd number $m < n$: if $|G| = m$, then G is solvable.
(This is a competition!)

(7) How many elements of order 7 are in a simple group of order 168?

(8) Show that the group $(\mathbb{Q}, +)$ does not have:

   (a) a normal sequence with cyclic factors,

   (b) a composition sequence.

(9) Find a composition sequence of the group $\mathbb{Z}_n$.

(10) Watch the following video about a certain simple group (not the Monster): http://www.youtube.com/watch?v=UTby_e4-Rhg.

(11) Show that if B is a set of free generators of G, then $\langle B\rangle = G$.
(You can use *only* the definition of a set of free generators!)

(12) Show that:
$$S_3 \cong \langle x, y | x^2 = y^3 = xyxy = 1\rangle.$$

(13) Show that:
$$D_n \cong \langle x, y | x^2 = y^n = xyxy = 1\rangle.$$

This is the link for Earliest Known Uses of Some of the Words of Mathematics: https://mathshistory.st-andrews.ac.uk/Miller/mathword/.