ALGEBRA 1R, Problem List 8

Let $(A, +)$ be a commutative group and p be a prime number.

(1) Assume that A is a subgroup of $\mathbb{Z}$. Show that $A = \{0\}$ or $A \cong \mathbb{Z}$.
(2) Find the product of cyclic groups which is isomorphic to the following group:
$$\mathbb{Z}^3/\langle(10, 11, 8), (4, 7, 4), (4, 4, 4)\rangle.$$
(3) Assume that $a_1, b_1, \ldots, a_k, b_k \in \mathbb{N}$ and we have:
$$\mathbb{Z}_p^{a_1} \times \mathbb{Z}_{p^2}^{a_2} \times \ldots \times \mathbb{Z}_{p^k}^{a_k} \cong \mathbb{Z}_p^{b_1} \times \mathbb{Z}_{p^2}^{b_2} \times \ldots \times \mathbb{Z}_{p^k}^{b_k}.$$
Show that:
$$a_1 = b_1, \ldots, a_k = b_k.$$
(Recall that $G^0 = \{e\}$.)
(4) Check whether the following groups are isomorphic:
 (a) $\mathbb{Z}_{24} \times \mathbb{Z}_{36}$ and $\mathbb{Z}_{48} \times \mathbb{Z}_{18}$,
 (b) $\mathbb{Z}_{21} \times \mathbb{Z}_{40}$ and $\mathbb{Z}_{168} \times \mathbb{Z}_5$,
 (c) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ and $\mathbb{Z}_{315}$.
(5) Assume that G is a finite group such that for all $g \in G$, we have $g^2 = e$. Show that there is $l \in \mathbb{N}$ such that:
$$G \cong (\mathbb{Z}_2)^l.$$
(6) Assume that $A = \{a_1, \ldots, a_n\}$ and that $\{b_1, \ldots, b_m\} \subset A$ consists of all elements of A of order 2.
 (a) Show the following:
$$a_1 + \ldots + a_n = b_1 + \ldots + b_m.$$
 (If $m = 0$, then we define the right-hand side above as 0.)
 (b) Show that $p - 1$ is a unique element of order 2 in the group $\mathbb{Z}_p^*$.
 (c) Conclude Wilson's Theorem.
(7) Assume that $|A| = n$ (A is still commutative!) and $k|n$. Show that there is $H \leqslant A$ such that $|H| = k$.