ALGEBRA 1R, Problem List 6

Let G be a group and $n \in \mathbb{N}_{>1}$.

(1) Let $g \in G$ be of order n. Show that for each $m \in \mathbb{Z}$, we have:
$$g^m = e \quad \text{if and only if} \quad n|m.$$
(2) Let $N \leqslant G$. Show that the following conditions are equivalent.
  (a) $N \trianglelefteqslant G$.
  (b) For each $g \in G$, we have $gNg^{-1} = N$.
  (c) For each $g \in G$ and for each $n \in N$, we have $gng^{-1} \in N$.
(3) Let $g \in G$ be of order 2. Show that the following conditions are equivalent:
  (a) $g \in Z(G)$,
  (b) $\{e, g\} \trianglelefteqslant G$.
(4) Let $g \in G$ be a unique element of order 2 in G. Show that $g \in Z(G)$.
(5) Using the fundamental theorem on group homomorphisms, show the following.
  (a) We have:
$$(\mathbb{R}^*, \cdot)/\{1, -1\} \cong (\mathbb{R}_{>0}, \cdot),$$
  (b) We have:
$$(\mathbb{C}, +)/\mathbb{Z} \cong (\mathbb{C}^*, \cdot),$$
  (c) We have:
$$(\mathbb{C}^*, \cdot)/\langle e^{\frac{2\pi i}{n}} \rangle \cong (\mathbb{C}^*, \cdot).$$
(6) Let p be a prime number and assume that $|G| = p^2$. Show that:
$$G \cong \mathbb{Z}_{p^2} \quad \text{or} \quad G \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$
(7) Let $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ be the group action from Problem 2 of List 5. Show that:
$$D_n \cong \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2.$$
(8) Show that:
$$A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3.$$
(9) Let G be the subset of $S_{\mathbb{R}}$ containing of *affine* bijections, that is maps of the form:
$$\mathbb{R} \ni x \mapsto ax + b \in \mathbb{R}.$$
Show the following.
  (a) $G \leqslant S_{\mathbb{R}}$.
  (b) We have:
$$G \cong (\mathbb{R}, +) \rtimes (\mathbb{R}^*, \cdot).$$
(10) Formulate and show a generalization of the statement in the previous problem from the case of $\mathbb{R}$ to the case of $\mathbb{R}^n$.
(11) Show that A_4 has no subgroup of order 6.