ALGEBRA 1R, Problem List 4

Let G be a group and $n \in \mathbb{N}_{>0}$.

(1) Show that if for all $g \in G$ we have $g^2 = e$, then G is commutative.

(2) Show that if $\sigma, \tau \in S_n$ are disjoint, then

$$\sigma \circ \tau = \tau \circ \sigma, \qquad X_{\sigma\circ\tau} = X_\sigma \cup X_\tau,$$

where X_σ denotes the support of the permutation σ.

(3) Show that if $n \geqslant 2$, then we have:

$$S_n = \langle (12), (12\ldots n)\rangle.$$

(4) Show that if $n \geqslant 3$, then we have:

$$A_n = \langle \{\sigma \in S_n \mid \sigma \text{ is a cycle of length } 3\}\rangle.$$

(5) Show that:

$$(\mathbb{Z}_2, +_2) \times (\mathbb{Z}_3, +_3) \cong (\mathbb{Z}_6, +_6).$$

How to generalize this result?

(6) Show that:

$$(\mathbb{Z}, +) \times (\mathbb{Z}, +) \ncong (\mathbb{Z}, +),$$
$$(\mathbb{Q}, +) \times (\mathbb{Q}, +) \ncong (\mathbb{Q}, +).$$

(7) Show that:

(a) For each $k \in \mathbb{Z}_n$, the function

$$\phi_k : (\mathbb{Z}_n, +_n) \to (\mathbb{Z}_n, +_n), \quad \phi_k(x) = k \cdot_n x$$

is an endomorphism.

(b) If

$$\phi : (\mathbb{Z}_n, +_n) \to (\mathbb{Z}_n, +_n)$$

is an endomorphism, then there is $k \in \mathbb{Z}_n$ such that $\phi = \phi_k$.

(c) If $k, l \in \mathbb{Z}_n$, then

$$\phi_k \circ \phi_l = \phi_{k\cdot_n l}.$$

(d) If $k \in \mathbb{Z}_n^*$, then $\phi_k \in \mathrm{Aut}(\mathbb{Z}_n, +_n)$.

(e) The function

$$\Phi : \mathbb{Z}_n^* \to \mathrm{Aut}(\mathbb{Z}_n, +_n), \quad \Phi(k) = \phi_k$$

is an isomorphism.