ALGEBRA 1R, Problem List 3

Let G and H be groups and $n \in \mathbb{N}_{>0}$.

(1) Show that if $H < G$ and G is cyclic, then H is cyclic.
(2) Assume that H_1 and H_2 are subgroups of G. Show that $H_1 \cup H_2$ is a subgroup of G if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.
(3) Is H a subgroup of G in the following cases?
  (a) $G = (\mathbb{Z}_8^*, \cdot_8)$, $H = \{1, 3\}$.
  (b) $G = (\mathbb{Z}_4, +_4)$, $H = \{0, 3\}$.
  (c) $G = (\mathbb{Z}_6, +_6)$, $H = \{0, 3\}$.
  (d) $G = D_4$ and H contains of all axial symmetries and the identity.
(4) Let $f : G \to H$ be a homomorphism, $G_1 \leqslant G$, and $H_1 \leqslant H$. Show that $f(G_1) \leqslant H$ and $f^{-1}(H_1) \leqslant G$.
(5) Show that the function

$$f : G \to G, \quad f(g) = g^2$$

is a homomorphism if and only if G is commutative.
(6) Let $g \in G$. We declare that $\min(\emptyset) = \infty$. Show that:

$$\mathrm{ord}(g) = \min\{k \in \mathbb{N}_{>0} \mid g^k = 1\}.$$

(7) Assume that $f : G \to H$ is a homomorphism, $g \in G$, and show the following.
  (a) If $\mathrm{ord}_G(g)$ is finite, then $\mathrm{ord}_H(f(g))$ is finite and

$$\mathrm{ord}_H(f(g)) | \mathrm{ord}_G(g).$$

  (b) If f is a monomorphism, then we have:

$$\mathrm{ord}_G(g) = \mathrm{ord}_H(f(g)).$$

(8) Assume that $G = \langle g \rangle$ is a cyclic group, H is an arbitrary group, and $h \in H$. Show the following.
  (a) If $\mathrm{ord}_G(g)$ is infinite, then there is a unique homomorphism $f : G \to H$ such that $f(g) = h$.
  (b) If $\mathrm{ord}_G(g)$ is finite, $\mathrm{ord}_H(h)$ is finite, and $\mathrm{ord}_H(h) | \mathrm{ord}_G(g)$, then there is a unique homomorphism $f : G \to H$ such that $f(g) = h$.
(9) Show that there is no monomorphism

$$(\mathbb{Z}_3, +_3) \to (\mathbb{R} \setminus \{0\}, \cdot).$$

(10) Show that $D_6 \not\cong A_4$.
(11) Show that $S^1 \cong \mathrm{SO}_2(\mathbb{R})$.
(12) Find a monomorphism

$$(\mathbb{Z}_n, +_n) \to (\mathbb{C} \setminus \{0\}, \cdot).$$

(13) Find a monomorphism

$$S_n \to \mathrm{GL}_n(\mathbb{Q}).$$

(14) Find all homomorphisms $f : G \to H$ in the following cases.
  (a) $G = (\mathbb{Z}_{15}, +_{15})$, $H = (\mathbb{Z}_4, +_4)$.
  (b) $G = (\mathbb{Z}_6, +_6)$, $H = (\mathbb{Z}_4, +_4)$.
  (c) $G = (\mathbb{Z}, +)$, $H = (\mathbb{Q}, +)$.
(15) Show that the group $(\mathbb{Q}, +)$ is not finitely generated.