ALGEBRA 1R, Problem List 2

Problem Session 13.10.2022 (Thursday).

Let $n \in \mathbb{N}_{>0}$ and G, H, N be groups. If $g \in G$, then $g^{-n} := (g^n)^{-1}$.

(1) Let us define:

$$\mathbb{Z}_n^* := \{a \in \mathbb{Z}_n \mid a \text{ is relatively prime with } n\}.$$

Show the following:

(a) The multiplication modulo n is associative (on $\mathbb{Z}_n$).

(b) For any $x, y \in \mathbb{Z}_n^*$, we have that $x \cdot_n y \in \mathbb{Z}_n^*$, so the multiplication modulo n can be considered as an operation on the set $\mathbb{Z}_n^*$.

(c) $(\mathbb{Z}_n^*, \cdot_n)$ is a group.

(2) Draw the multiplication table of D_3.

(3) Draw the multiplication table of the group of isometries of a rectangle which is not a square.

(4) For $k, l \in \mathbb{Z}$ and $g \in G$ show that

$$g^k g^l = g^{k+l}, \quad (g^k)^l = g^{kl}.$$

(Consider all the cases, for example: $k < 0, l < 0$ or $k < 0, l > 0$!)

(5) Let $(A, +)$ be a commutative group and $m \in \mathbb{Z}$. Show that the function

$$f : A \longrightarrow A, \quad f(a) := ma$$

is a homomorphism.

(6) Find an isomorphism between D_3 and S_3.

(7) Find a monomorphism

$$(\mathbb{R}, +) \longrightarrow \mathrm{GL}_2(\mathbb{R}).$$

(8) Find a monomorphism

$$(\mathbb{Z}_n, +_n) \longrightarrow D_n.$$

(9) Let G be the group from Problem 3. Show that G is not isomorphic with the group $(\mathbb{Z}_4, +_4)$.

(10) Find a monomorphism

$$(\mathbb{C} \setminus \{0\}, \cdot) \longrightarrow \mathrm{GL}_2(\mathbb{R}).$$

(11) Show that if $f : G \to H$ is a homomorphism and $g \in G$, then for all $m \in \mathbb{Z}$ we have:

$$f(g^m) = f(g)^m.$$

(12) Show the following:

(a) The composition of two homomorphisms is a homomorphism,

(b) The inverse function to an isomorphism is an isomorphism,

(c) If $G \cong H$ and $H \cong N$, then $G \cong N$.

(13) Let X and Y be sets such that $|X| = |Y|$. Show that

$$S_X \cong S_Y.$$