ALGEBRA 1R, Problem List 13

Let R and S be commutative rings with 1.

(1) For each prime number p, we identify $\mathbb{Z}_{(p)}$ with the corresponding subring of $\mathbb{Q}$. Show the following:
$$\bigcap_p \mathbb{Z}_{(p)} = \mathbb{Z}.$$

(2) Assume that R is UFD and let $f, g \in R[X]$. Show that if $\mathrm{cont}(f) = 1$ and $\mathrm{cont}(g) = 1$, then $\mathrm{cont}(fg) = 1$.
Recall that we have:
$$\mathrm{cont}\left(a_0 + a_1X + \ldots + a_nX^n\right) = 1$$
if and only if there is no irreducible element $p \in R$ such that:
$$p|a_0, p|a_1, \ldots, p|a_n.$$

(3) Show that $X^{p-1} + X^{p-2} + \cdots + X + 1 \in \mathbb{Q}[X]$ is irreducible, where p is a prime number.

(4) Find a greatest common divisor and a least common multiple for:
  (a) $X^4 - X,\ X^6 - X$ in $\mathbb{C}[X]$,
  (b) $X^4 - X,\ X^6 - X$ in $\mathbb{C}[\![X]\!]$,
  (c) $4 - 2i,\ 13 + i$ in $\mathbb{Z}[i]$,
  (d) $13,\ 12 + 5i$ in $\mathbb{Z}[i]$,

(5) Let
$$R := \{a_0 + 2a_1X + \ldots + 2a_nX^n \in \mathbb{Z}[X] \mid a_0, a_1, \ldots, a_n \in \mathbb{Z};\ n \in \mathbb{N}\}.$$
Show the following:
  (a) R is a subring of $\mathbb{Z}[X]$;
  (b) the ideal $(2X) \cap (2X^2)$ is not principal in R;
  (c) the elements $2X$ and $2X^2$ have no least common multiple in R.

(6) Show that the following conditions are equivalent:
  (a) There are R_1, R_2, non-zero rings with 1 such that
  $$R \cong R_1 \times R_2.$$
  (b) There are $u_1, u_2 \in R \setminus \{0\}$ such that
  $$u_1 + u_2 = 1, \quad u_1^2 = u_1, \quad u_2^2 = u_2.$$
  (c) There is $u \in R \setminus \{0, 1\}$, which is an *idempotent*, that is $u^2 = u$.

(7) Show that
$$(R \times S)^* = R^* \times S^*,$$
which is an equality of subsets of $R \times S$.

(8) Let $n \in \mathbb{N}$ and $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, where $\alpha_i \in \mathbb{N}$ and $p_1, \ldots, p_k$ are pairwise distinct prime numbers. Show the following:
  (a) For $\alpha \in \mathbb{N}$ and a prime number p, we have:
  $$|\mathbb{Z}_{p^\alpha}^*| = p^\alpha - p^{\alpha-1}.$$
  (b) We have:
  $$|\mathbb{Z}_n^*| = \left(p_1^{\alpha_1} - p_1^{\alpha_1-1}\right) \cdot \ldots \cdot \left(p_k^{\alpha_k} - p_k^{\alpha_k-1}\right).$$

(9) Show that:
$$\mathbb{Q}[X,Y]/(XY) \not\cong \mathbb{Q}[X,Y]/(X) \times \mathbb{Q}[X,Y]/(Y).$$