ALGEBRA 1R, Problem List 11

Let R be a commutative ring with 1 and $n \in \mathbb{N}_{>0}$.

(1) Let $r_1, \ldots, r_n \in R$. Show that:
$$(r_1, \ldots, r_n) = r_1R + \ldots + r_nR.$$

(2) Let $I \trianglelefteq R$ and
$$\sqrt{I} := \{a \in R : (\exists n \in \mathbb{N})(a^n \in I)\}.$$
Show that $\sqrt{I} \trianglelefteq R$.

(3) Let $f : R \to S$ be a homomorphism of commutative rings with 1, $I \trianglelefteq R$, and $J \trianglelefteq S$. Show the following:
- $f^{-1}(J) \trianglelefteq R$.
- If f is an epimorphism, then $f(I) \trianglelefteq S$.
- Give an example of f, I such that $f(I) \ntrianglelefteq S$.

(4) Find $f \in \mathbb{Q}[X]$ such that
$$(f) = (X^2 - 1, X^3 + 1).$$

(5) Show that the ideal $(2, X) \trianglelefteq \mathbb{Z}[X]$ is not principal.

(6) Let $\phi : R \to S$ be an epimorphism of rings, where R is Noetherian. Show that S is Noetherian as well.

(7) Find a subring $R \subseteq \mathbb{Z}[X]$ such that R is not Noetherian.

(8) Show that the ring $\mathbb{Z}[\sqrt{2}]$ is Euclidean.

(9) Let $d \in \mathbb{C} \setminus \mathbb{Z}$ such that $d^2 \in \mathbb{Z}$. We define:
$$v : \mathbb{Q}(d) \to \mathbb{Q}, \quad v(n + md) = n^2 - m^2d^2.$$
Show the following.
   (a) For each $\alpha, \beta \in \mathbb{Q}(d)$, we have
   $$v(\alpha\beta) = v(\alpha)v(\beta).$$
   (b) For each $\alpha \in \mathbb{Z}[d]$, we have $\alpha \in \mathbb{Z}[d]^*$ if and only if $v(\alpha) \in \{-1, 1\}$.

(10) Describe the group $\mathbb{Z}[\sqrt{-5}]^*$.

(11) Show that the group $\mathbb{Z}[\sqrt{2}]^*$ is infinite.