BOREL REDUCIBILITY
AND
HIGHER SET THEORY

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Borel reducibility.

Definition. Let E, F be analytic equivalence relations on X, Y. Say that $E \leq F$ if there is a Borel function $h : X \to Y$ which is a reduction $\forall x_0, x_1 \in X \;\; x_0 \; E \; x_1 \leftrightarrow h(x_0) \; F \; h(x_1)$.

Example. E_0 on 2^ω, connecting x, y if $x \Delta y$ is finite.

Example. F_2 on $(2^\omega)^\omega$, connecting x, y if $\mathrm{rng}(x) = \mathrm{rng}(y)$.

Example. E_{K_σ} on ω^ω connecting x, y if $x - y$ is bounded.

Main challenge.

Proving that E is not reducible to F

Definition. A property Φ of equivalence relations is a Borel reducibility invariant if $\Phi(F)$ and $E \leq F$ implies $\Phi(E)$.

A dream. Finding reducibility invariants that connect to higher stages of the cumulative hierarchy, similar to Shelah's classification of models.

Forcing.

Interpretations. Every Polish space and analytic subset of a Polish space have canonical interpretation in any generic extension. Denoted by same letter.

Shoenfield absoluteness. Every $\mathbf{\Pi}^1_2$ sentence is absolute between forcing extensions.

Example. If E, F are analytic equivalence relations on X, Y and $h : X \to Y$ is a function, the statement "h is a reduction" is $\mathbf{\Pi}^1_2$ and therefore absolute.

Example. If E, F are Borel then the statement $E \leq F$ is $\mathbf{\Pi}^1_2$ and therefore absolute.

Pinned names and equivalence relations.

Definition. (Kanovei) Let E be an analytic equivalence relation on X. A P-name τ for an element of X is E-*pinned* if $P \times P \Vdash \tau_l \, E \, \tau_r$.

Definition. τ is E-*trivial* if there is a point $x \in X$ such that $P \Vdash \tau \, E \, \dot{x}$.

Definition. E is *pinned* if every E-pinned name on every poset is trivial.

Main fact. "E is pinned" is a reducibility invariant.

Example. F_2 is not pinned.

Proof. Let P be the collapse of 2^ω to ω. Let $A \subset 2^\omega$ be uncountable and let τ be a P-name for an enumeration of A in the generic extension in ordertype ω. This is a nontrivial pinned name.

Remark. All pinned names for F_2 can be represented in this way.

Question. (Kechris) Is F_2 minimal among the unpinned equivalence relations?

Example. E_{K_σ} is pinned.

Proof. Let P be a poset and τ a pinned E_{K_σ}-name. There is a condition $p \in P$ and a number n such that for every $m \in \omega$ the possibilities for $\tau(m)$ are at most n far apart below p. Let $x \in \omega^\omega$ be a point defined by $x(m) =$the least number k such that there is $q \leq p$ with $q \Vdash \tau(\check{m}) = \check{k}$. Then $p \Vdash \tau \ E_{K_\sigma} \ \check{x}$ and τ is pinned.

Remark. E_{K_σ} is not the $\leq$-largest pinned equivalence relation.

Question. Is there a-$\leq$-largest pinned equivalence relation?

Other features of pinned equivalence relations

Theorem. The following are equivalent for an analytic equivalence relation E:

- E is unpinned;
- E is unpinned in all generic extensions;
- there is a nontrivial pinned name in every poset collapsing $\aleph_1$ to $\aleph_0$.

If E is Borel on X then the set $\{A \subset X : E \restriction A$ is pinned$\}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

A partial dichotomy.

Theorem. Let κ be a measurable cardinal. Let W be the Solovay choiceless model derived from κ. The following are equivalent for any Borel equivalence relation E in W:

- E is unpinned;
- $F_2 \leq E$.

For analytic equivalence relation E, $E_{\omega_1} \leq_w E$ must be added to the second item.

Equivalence relation on names.

Definition. Let E be an equivalence relation on a Polish space X. Let P, Q be posets and τ, σ be P, Q-names for elements of X. Say that $\langle P, \tau \rangle \; \bar{E} \; \langle Q, \alpha \rangle$ if $P \times Q \Vdash \tau \; E \; \sigma$.

Fact. $\bar{E}$ is an equivalence relation on pinned names.

Question. How many $\bar{E}$ classes are there? On which posets do they live?

The pinned cardinal

Definition. Let E be an analytic equivalence relation on a Polish space X. The pinned cardinal $\kappa(E)$ is the smallest κ such that every pinned name τ has a $\bar{E}$-equivalent on a poset of size $< \kappa$.

- $\kappa(E)$ can be equal to ∞;
- $\kappa(E) = \aleph_1$ if E is pinned as a definitory matter.

Fact. if $E \leq F$ then $\kappa(E) \leq \kappa(F)$.

Basic features of the pinned cardinal.

Bounds:

- $\kappa(E) < \beth_{\omega_1}$ if E is Borel;

- $\kappa(E) <$the first measurable if $\kappa(E) < \infty$;

- $\kappa(E) = \infty$ iff E_{ω_1} is weakly reducible to E.

Operations:

- $\kappa(E^+) \leq (2^{<\kappa(E)})^+$;

- $\kappa(\prod_I E_n) \leq \max(\kappa(E_n))$ whenever I is a Borel ideal on ω and $=_I$ is pinned.

Examples.

- $\kappa(F_\alpha) = \beth_\alpha^+$;
- there are Borel equivalence relations E_α for each countable $\alpha > 0$ such that provably $\kappa(E_\alpha) = \aleph_\alpha$;
- there is Borel E such that provably $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$;
- there is analytic E such that under MA, $\kappa(E) = \aleph_2$ iff Chang's conjecture holds;

Question. What is the pinned cardinal of the measure equivalence?

The $\aleph_\alpha$ example.

Definition. A $L_{\omega_1\omega}$-sentence ϕ is *set-like* if there is a binary relation e such that ϕ proves that e is extensional and well-founded.

Theorem. Let ϕ be set-like and E_ϕ be the isomorphism of countable models of ϕ. Then $\kappa(E_\phi)$ =the least cardinal κ such that ϕ has no model of size κ.

Exercise. By induction on $0 \neq \alpha \in \omega_1$ build sentences ϕ_α such that the sentence has models of all sizes $< \aleph_\alpha$ but no model of size $\aleph_\alpha$.

To ϕ_α, add a sentence saying "e is an extensional well-founded relation of rank α" . This does not change the cardinal, since $|V_{\omega+\alpha}| \geq \aleph_\alpha$.

The cardinal arithmetic example.

1. Find sentences ϕ, ψ such that ϕ has models of size $\leq \aleph_\omega$ and ψ has models of size $\leq \aleph_0$. Let θ be the sentence "$A \models \phi$, $B \models \psi$, and C is a collection of pairwise distinct maps from B to A". Add an extensional well-founded relation of rank $\leq \omega + \omega + 2$. Then $\kappa(E_\theta) = (\aleph_\omega^{\aleph_0})^+$.

2. Find sentences ϕ, ψ such that ϕ has models of size $\leq \aleph_{\omega+1}$ and ψ has models of size $\leq \mathfrak{c}$. Let $\nu = \psi \vee \phi$, and add an extensional well-founded relation of rank $\leq \omega + \omega + 2$. Then $\kappa(E_\nu) = \max(\aleph_{\omega+1}, \mathfrak{c})^+$.

3. E_θ is not Borel reducible to E_ν since SCH fails in a forcing extension and so $\kappa(E_\theta) < \kappa(E_\nu)$ is consistent.

Chang Conjecture Example.

Fact. Under MA, CC is equivalent to $\binom{\omega_2}{\omega_2} \to \binom{\omega}{\omega}^1_\omega$.

Let $f : X \times X \to \omega$ be a universal F_σ-function. Let $A \subset X^\omega$ be the coanalytic set of y such that rng$(y)\times$rng(y) contains no infinite rectangle on which f is constant. Let E be $(F_2 \restriction A)\cup(\tilde{A}\times\tilde{A})$. This is an analytic equivalence relation.

If MA and CC holds then no pinned name τ can have $|\text{rng}(\tau)|^V \geq \aleph_2$ and so $\kappa(E) = \aleph_2$. If CC fails, let $g : \omega_2 \times \omega_2 \to \omega$ be a counterexample to the partition, let $h : \omega_2 \to X$ be such that $g = f \circ h$, and let τ be a Coll(ω, ω_2)-name for rng(h). So $\kappa(E) > \aleph_2$.

Pinned names and posets.

Fact. Nontrivial pinned names cannot live on proper posets. They mostly live on collapse posets.

Example. If τ is an F_2-pinned name then there is a set $A \subset 2^\omega$ such that $\Vdash \text{rng}(\tau) = \check{A}$.

Definition. Let E be an analytic equivalence relation on X and τ an E-pinned name on some P. $\kappa(\tau)$ is the unique cardinal κ if it exists such that a poset Q carries an $\bar{E}$-equivalent to τ if and only if Q collapses κ to $\aleph_0$.

Definition. E is cardinalistic if $\kappa(\tau)$ exists for every E-pinned name τ.

Examples of cardinalistic relations.

Theorem. Every orbit equivalence relation is cardinalistic.

Theorem. The cardinalistic property is preserved under Friedman–Stanley jump, product modulo an ideal J such that $=_J$ is pinned.

Fact. To be cardinalistic is a reducibility invariant.

Nonexamples of cardinalistic relations.

Definition. The mutual domination equivalence relation E on $X = (\omega^\omega)^\omega$ connects points x, y if for every n there is m such that $x(n)$ is modulo finite dominated by $y(m)$ and vice versa.

Theorem. If there is a modulo finite increasing ω_2-sequence of points in ω^ω, then E is not cardinalistic.

Proof. Let $\langle z_\alpha : \alpha \in \omega_2 \rangle$ be the increasing sequence, let P be the Namba forcing adding a cofinal sequence $\langle \alpha_n : n \in \omega \rangle$, and let $\tau = \langle x_{\alpha_n} : n \in \omega \rangle$.

Hjorth's turbulence.

Definition. Let E, F be analytic equivalence relations on X, Y. E is F-*ergodic* if for every Borel homomorphism $h : X \to Y$, there is an F-class with comeager h-preimage.

- turbulence applies to orbit equivalence relations only–dynamic concept;

- turbulent equivalences are F-ergodic for every F classifiable by countable structures;

- useful only in Baire category context.

Turbulence generalized.

- independent of any dynamical context;
- compares certain generic extensions;
- ergodicity results;
- can be tailored to measure context etc.

Motivating restatement of turbulence.

Definition. Let G act on X continuously. Let P_G be the Cohen poset on G, with generic $\dot{g}$. Let P_X be the Cohen poset on X, with generic $\dot{x}$.

Theorem. The following are equivalent:

1. the action is generically turbulent;

2. $P_G \times P_X \Vdash V[\dot{x}] \cap V[\dot{g} \cdot \dot{x}] = V$.

Trimness and variations.

Definition. Let E be an analytic equivalence relation on X. E is *trim* if for every pair $V[G_0], V[G_1]$ of generic extensions containing respective E-related points x_0, x_1, if $V[G_0] \cap V[G_1] = V$ then $x_0 \; E \; x$ for some $x \in V$.

Generalization I. $\mathfrak{P}$-trimness results from restricting the extensions to posets with property $\mathfrak{P}$.

Generalization II. ω-trimness results from considering infinitely many extensions.

Connection with pinned property.

Definition. Let τ, σ be P, Q-names respectively for elements of X. Say that $\langle P, \tau \rangle \ \hat{E} \ \langle Q, \sigma \rangle$ is for every $p \in P$ and every $q \in Q$ there are generic extensions $V[G], V[H]$ such that

- $p \in G \subset P$, $q \in H \subset Q$;
- $V[G] \cap V[H] = 0$;
- $tau/G \ E \ \sigma/H$.

Fact. $\hat{E}$ is an equivalence relation on trim names, an equivalence is trim if there are non-trivial trim names etc.

Main properties.

- all variants of trimness are reducibility invariants;

- trimness and proper-trimness are absolute between forcing extensions;

- trim is equivalent to the conjunction of pinned and proper-trim;

- proper-trim is equivalent to Cohen-trim.

Theorem. Let J be the ideal on $2^{<\omega}$ generated by branches. The relation $=_J$ is trim.

Proof. Let $x_0 \in V[G_0]$, $x_1 \in V[G_1]$ are $=_J$-related points. Assume $V[G_0] \cap V[G_1] = V$ and $x_0 \neq_J x$ for every $x \in V$. Work for contradiction.

Let $T = \{t \in 2^{<\omega} : x_0 \restriction [t] \neq_J y$ for any $y \in V\}$. This is a tree with no terminal nodes, containing 0. It is in $V[G_0] \cap V[G_1]$, so in V. Pick an infinite branch $b \in V$.

There is $t \in b$ such that $x_0 \restriction [t] = y_0 \restriction [t]$ off b. Then $x_0 \restriction [t]$ off b is in V, contradicting the definition of T.

Theorem. F_2 is proper-trim.

Proof. Suppose that $V[G_0]$, $V[G_1]$ are proper generic extensions containing F_2-related points x_0, x_1. Suppose that $V[G_0] \cap V[G_1] = V$. Work to find $x \in V$ with $x \; F_2 \; x_0$.

For every $z \in \mathrm{rng}(x_0) \cap \mathrm{rng}(x_1)$, $z \in V[G_0] \cap V[G_1] = V$. Thus, $\mathrm{rng}(x_0) = \mathrm{rng}(x_1)$ is a subset of V and an element of V.

Since $V[G_0]$ is proper, $\mathrm{rng}(x_0)$ is countable in V and so there is $x \in V$ such that $\mathrm{rng}(x) = \mathrm{rng}(x_0)$ and so $x \; F_2 \; x_0$ as required.

Theorem. Let J be the ideal on $A = \prod_{n\in\omega}(n+1)$ generated by graphs of functions. Then $=_J$ is not Cohen-trim.

Proof. Let P be the poset of partial finite functions from A to 2 ordered by inclusion, adding $\dot{x} \in 2^A$. Find generic filters $G_0, G_1 \subset P$ such that $\dot{x}/G_0 =_J \dot{x}/G_1$ and $V[G_0] \cap V[G_1] = 0$.

The poset Q adding the two generics consists of conditions $q = \langle p_0, p_1 \rangle$ where $p_0, p_1 \in P$ with the same domain such that $\{a \in \text{dom}(p_0) : p_0(a) \neq p_1(a)\}$ is a graph of a partial function.

Key: if $p \Vdash \tau \subset Ord$ is not in V, then there are extensions $p_0, p_1 \leq p$ which differ at just one entry and an ordinal α such that $p_0 \Vdash \check{\alpha} \in \tau$ and $p_1 \Vdash \check{\alpha} \notin \tau$.

Ergodicity in category

Definition. Let E, F be analytic equivalence relations on X, Y, and t a Polish topology on X. Say that E is F-t-ergodic if for every Borel homomorphism $h : X \to Y$ there is an F-class with t-comeager preimage.

Example. E_0 is id-t-ergodic where t is the usual topology on 2^ω.

Fact. Different topologies may give different ergodicity properties.

Theorem. If E is not proper-trim, then there is a Polish topology t on X with meager classes such that for every proper-trim F, E is F-t-ergodic.

Proof. Let τ be a nontrivial trim name on the Cohen poset. Let t be an associated Polish topology on X (B is meager if $P \Vdash \tau \notin B$). This topology works.

Suppose that $h : X \to Y$ is a Borel homomorphism of E to proper-trim F. Let $G, H \subset P$ be generic filters such that $\tau/G \ E \ \tau/H$ and $V[G] \cap V[H] = V$. Then $h(\tau/G) \ F \ h(\tau/H)$, and proper-trimness of F shows that there is $y \in Y \cap V$ such that $h(\tau/G) \ F \ y$. Then $h^{-1}[y]_F$ is t-comeager.

Corollaries.

Let J be the ideal on $2^{<\omega}$ generated by branches.

Corollary. If E is an orbit equivalence of a turbulent action, then E is $=_J$-ergodic. (E is not proper-trim while E is.)

Corollary. If F is an equivalence classifiable by countable structures then $=_J$ is F-ergodic. ($=_J$ is not Cohen-ω-trim while F is.)

Ergodicity in measure

Definition. Let E, F be analytic equivalence relations on X, Y, let μ be a Borel probability measure on X. Say that E is F-μ-ergodic if for every Borel homomorphism $h : X \to Y$ there is an F-class with preimage of full μ-mass.

Example. E_0 is id-μ-ergodic for the usual probability measure μ on 2^ω.

Fact. Ergodicity in measure in K_σ equivalence relations closely relates to concentration of measure.

Concentration of measure.

Definition. A sequence $\langle X_n, d_n, \mu_n, \varepsilon_n, \delta_n \rangle$ has *concentration of measure* if

- X_n is a (finite) set with metric d_n and probability measure μ_n;

- $\varepsilon_n, \delta_n > 0$ are real numbers tending to zero;

- for every set $A \subset X_n$ of μ_n-mass $> \delta_n$, the ε_n-neighborhood of A has μ_n-mass $> 1/2$.

Fact. Many examples such as n-dimensional spheres with the usual measure and metric.

Theorem. Let $\langle X_n, d_n, \mu_n, \varepsilon_n, \delta_n \rangle$ have concentration of measure. Let

- $X = \prod_n X_n$,$\mu = \prod_n \mu_n$;
- $x \ E \ y$ if there is m such that for all n, $d_n(x(n), y(n)) \leq m\delta_n$.

Then E is F-μ-ergodic for every proper-trim equivalence relation F.

Proof. Add μ-random points $x, y \in X$ such that $\forall n\ d_n(x(n), y(n) \leq \delta$ and $V[x] \cap V[y] = V$.

The poset consists of all pairs $q = \langle B_q, s_q, t_q \rangle$ where $B_q \subset X$ is a Borel positive set, $s_q, t_q \in \prod_{n \in m} X_n$, for all $n \in m\ d_n(s_q(n), t_q(n)) \leq \delta$, and $B_q \subset [s_q]$.

The ordering is defined by $r \leq q$ if $B_r \subset B_q, s_q \subset s_r t_q \subset t_r$, and $B_r \cdot t_r \subset B_q \cdot t_q$. The names $\dot{x}, \dot{y}$ are defined as $\dot{x} = \bigcup_{q \in G} s_q$ and $\dot{y} = \bigcup_{q \in G} t_q$.

The fact that $V[x] \cap V[y] = V$ uses the concentration of measure assumption.

Corollary. E_2 is F-μ-ergodic for every equivalence relation F classifiable by countable structures.

Here, the equivalence relation E_2 on $X = 2^\omega$ relates points x, y if $\sum\{1/n+1 : x(n) \neq y(n)\} < \infty$. The measure μ is the usual measure on the space 2^ω.

Separation properties of equivalence relations.

Fact. (Martin–Solovay) If MA for κ holds and $A_0, A_1 \subset X$ are disjoint sets of size $\leq \kappa$, then there is a K_σ set B_0 such that $A_0 \subset B_0$ and $A_1 \cap B_0 = 0$.

Question. Does something similar hold for quotient spaces X/E for various equivalence relations E?

Fact. Possessing a separation property is a reducibility invariant.

A correct generalization of Martin–Solovay.

Main tool. If MA σ-centered for κ holds, $G \subset X^n$ is an F_σ set and $A \subset X$ is G-free, then there is a K_σ set B such that $A \subset B$ and B is G-free.

Example. To separate disjoint sets $A_0, A_1 \subset X$, apply the tool to $Y = X \times 2$, G connecting distinct points with the same X coordinate, and $A = (A_0 \times \{0\}) \cup (A_1 \times \{1\})$.

Fact. In ZFC, there is a G_δ-graph $G \subset X^2$ and a set $A \subset X$ of size $\aleph_1$ which is G-free and every analytic G-free set has countable intersection with A.

Where separation holds.

Theorem. If Martin's Axiom for σ-centered holds at κ, then the separation property holds for E_{K_σ} at κ.

Proof. Let $A_0, A_1 \subset \omega^\omega$ be E_{K_σ}-unrelated sets of size $\leq \kappa$. Apply the tool to $Y = X \times 2$, G connecting distinct points with E_{K_σ}-equivalent X coordinates, and $A = (A_0 \times \{0\}) \cup (A_1 \times \{1\})$.

Question. Let J be an analytic P-ideal on ω. Does separation hold for $=_J$?

Where separation fails.

Theorem. There is a sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ of pairwise F_2-inequivalent points of $X = (2^\omega)^\omega$ such that no analytic F_2-invariant set $A \subset X$ cuts out a stationary-costationary piece.

Proof. Let x_α be points such that the sets $\mathrm{rng}(x_\alpha) \subset 2^\omega$ continuously increase with respect to inclusion. Let $A \subset X$ be an analytic F_2-invariant set. Let $P = \mathrm{Coll}(\omega, \omega_1)$, let τ be a name for an element of X enumerating $\bigcup_\alpha \mathrm{rng}(x_\alpha)$ and let p decide the statement $\tau \in \dot{A}$.

If $p \Vdash \tau \in \dot{A}$ then $\{\alpha : x_\alpha \in A\}$ contains a club, if $p \Vdash \tau \notin \dot{A}$ then $\{\alpha : x_\alpha \notin A\}$ contains a club.

I-sequences.

Definition. Let E be an analytic equivalence relation on X, I a σ-ideal on a cardinal κ. A sequence $\langle x_\alpha : \alpha \in \kappa \rangle$ is an I-sequence if for every analytic E-invariant set A, $\{\alpha : x_\alpha \in A\} \in I$ or $\{\alpha : x_\alpha \notin A\} \in I$.

Fact. Not having an I-sequence is a reducibility invariant.

Theorem. Every unpinned equivalence relation has an I-sequence for $I =$the nonstationary ideal on ω_1.

Proof. Let E be an unpinned equivalence relation on a space X. Let τ be a nontrivial pinned P-name for E, where $|P| = \aleph_1$.

Let $\langle M_\alpha : \alpha \in \omega_1 \rangle$ be an $\in$-tower of continuous elementary submodels. Let $x_\alpha \in X$ represent the unique E-class of τ/g where $g \subset M_\alpha \cap P$ is generic. $\langle x_\alpha : \alpha \in \omega_1 \rangle$ is an I-sequence.

Suppose that $A \subset X$ is analytic E-invariant and $p \Vdash \tau \in \dot{A}$. Whenever N is a countable elementary submodel, there is $\alpha \in \omega_1$ such that $P \cap N = P \cap M_\alpha$. Then $x_\alpha \in A$.

Theorem. (MA) If J is an analytic P-ideal then $=_J$ has no I-sequence for a normal σ-ideal I.

Proof. Find a submeasure μ on ω such that $J = \{a \subset \omega : \lim_n \mu(a \setminus n) = 0\}$. Let $d(x, y) = \lim_n \mu(x \Delta y \setminus n)$ for $x, y \in X = \mathcal{P}(\omega)$. Work with $I =$the nonstationary ideal on ω_1. Let $\langle x_\alpha : \alpha \in \omega_1 \rangle$ be a sequence.

Either there is $x \in X$ and $\varepsilon > 0$ such that $\{\alpha : d(x_\alpha, x) < \varepsilon\}$ is stationary costationary.

Or, there is a club C and $\varepsilon > 0$ such that $d(x_\alpha, x_\beta) > \varepsilon$ if $\alpha \neq \beta \in C$. Let $C = S_0 \cup S_1$ be a partition into stationary sets. Use the graph theorem to find a K_σ-set $A \subset X$ such that $d(x_\alpha, A) \geq \varepsilon$ if $\alpha \in S_0$ and $x_\beta \in A$ if $\beta \in S_1$.

Let $B = \{x \in X : d(x, A) \geq \varepsilon\}$.

Theorem. (MA) If E is classifiable by countable structures, then E has no I-sequence for any normal $< \aleph_2$-complete ideal.

Proof. Deal with $E = F_2$, $I =$nonstationary ideal on ω_2. Let $\langle x_\alpha : \alpha \in \omega_2 \rangle$ be a sequence. Let $a = \{z \in 2^\omega : \{\alpha : z \in \text{rng}(x_\alpha)\}$ contains a club$\}$. The set $a \subset 2^\omega$ is countable.

Use normality to pick points $\langle z_\alpha : \alpha \in C \rangle$ such that $z_\alpha \in \text{rng}(x_\alpha) \setminus \bigcup_{\beta \in \alpha} \text{rng}(x_\beta)$. Let $C = S_0 \cup S_1$ be a partition into stationary sets. Use the graph theorem to find a K_σ-set $A \subset 2^\omega$ such that $A \cap \text{rng}(x_\alpha) = 0$ if $\alpha \in S_0$ and $A \cap \text{rng}(x_\alpha) = \{z_\alpha\}$ if $\alpha \in S_1$.

Let $B = \{x \in X : \text{rng}(x) \cap A = 0\}$.

Theorem. (MA) Let E be the mutual domination equivalence. E has an I-sequence where I is the nonstationary ideal on ω_2 restricted to $\mathtt{cof}(\omega)$.

Proof. Let $\langle z_\alpha : \alpha \in \omega_2\rangle$ be a modulo finite increasing sequence in ω^ω. If $\mathtt{cof}(\alpha) = \omega$ then let $x_\alpha = \langle z_{\alpha_n} : n \in \omega\rangle$ for $\alpha_n \to \alpha$. $\langle x_\alpha : \alpha \in \omega_2 \cap \mathtt{cof}(\omega)\rangle$ is an I-sequence.

If $A \subset X$ is an analytic E-invariant set, let $P =$ Coll(ω, ω_2), let τ be a P-name for $\langle z_{\alpha_n} : n \in \omega\rangle$ for $\alpha_n \to \omega_2^V$. If $p \Vdash \tau \in A$ then there is a club $C \subset \omega_2$ such that $\forall\alpha \in C\ x_\alpha \in A$.