Applications of infinitary combinatorics **8** 2019

Kunen's L-space [1981]

Explanations:

- HS = hereditarily separable,
- HL = hereditarily Lindelof,
- X - a Hausdorff space,
- K - the space constructed on the lecture,
- λ - the standard measure on 2^{ω_1}.

Zad. 1 Show that if X is metrizable, then X is HL $\iff$ X is HS.

Zad. 2 Show that if there is a continuous surjection $f\colon X \to [0,1]^{\omega_1}$, then there is a non-separable measure on X.

Zad. 3 Show that whenever $\mathfrak{A}$ is a countable subalgebra of Borel(2^{ω_1}), and B is a Borel subsets of 2^{ω_1} such that $\lambda(B) > 0$, then there is a closed $F \subseteq B$ such that

$$\inf\{\lambda(A \triangle F)\colon A \in \mathfrak{A}\} > 0.$$

Zad. 4 Let $\mathcal{F}$ be the family of subsets used to extend the Boolean algebras in the construction of the space K, i.e.

$$\mathcal{F} = \{F_\xi\colon \xi < \omega_1\} \cup \{F^\xi_\alpha\colon \xi, \alpha < \omega_1\}.$$

Show that for every $x \in K$

$$|\{F \in \mathcal{F}\colon x \in F\}| = \aleph_0.$$

Conclude that K is not separable (hint: otherwise K would have countable base). Hence, K is an L-space.

Zad. 5 Let $(x_\alpha)_{\alpha<\omega_1}$ be a well-ordered sequence of (different) real numbers. Denote $Z = \{x_\alpha\colon \alpha < \omega_1\}$.

- Let τ_s be the topology on Z defined by neighbourhoods of points in the following way: the neighbourhoods of x_α are of the form $(x_\alpha - \varepsilon, x_\alpha + \varepsilon) \cap \{x_\xi\colon \xi \leq \alpha\}$.
- Let τ_l be the topology on Z defined by neighbourhoods of points in the following way: the neighbourhoods of x_α are of the form $(x_\alpha - \varepsilon, x_\alpha + \varepsilon) \cap \{x_\xi\colon \xi \geq \alpha\}$.

Show that (Z, τ_s) is an S-space, and (Z, τ_l) is an L-space. Show that those spaces are Hausdorff but not regular.

Pbn
`http://www.math.uni.wroc.pl/~pborod/dydaktyka`