Applications of infinitary combinatorics **3** 2018

Stone spaces.

Zad. 1 Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and let $K = \mathrm{St}(\mathfrak{A})$ and $L = \mathrm{St}(\mathfrak{B})$. Show that

- $\mathfrak{A}$ can be embedded in $\mathfrak{B}$ (by a Boolean monomorphism) if and only if there is a continuous surjection $f\colon L \to K$.
- K can be embedded in L (by an injective homeomorphism) if and only if there is a Boolean epimorphism $h\colon \mathfrak{B} \to \mathfrak{A}$.

Zad. 2 Let K be a compact zerodimensional space. Show that K is homeomorphic to $\mathrm{St}(\mathrm{Clop}(K))$.

Zad. 3 Let $\mathfrak{A}$ be a Boolean algebra generated by an uncountable almost disjoint family $\mathcal{A}$ and the family of finite sets. How does it Stone space look like? What is the difference with Mrówka space generated by $\mathcal{A}$?

Zad. 4 What is $\mathrm{St}(\mathfrak{A})$ if $\mathfrak{A}$ is a Boolean algebra generated by $\mathcal{A}$ and the family of finite sets where:

- $\mathcal{A}$ is a tower?
- $\mathcal{A}$ is a gap?
- $\mathcal{A}$ is a density filter?

Zad. 5 Consider a Boolean algebra $\mathfrak{A}$ of subsets of $[0,1]$ generated by sets of the form $[0,a)$, where $a \in [0,1]$. What is the Stone space of $\mathfrak{A}$?

Zad. 6 We say that a Boolean algebra $\mathfrak{A}$ is σ-centered if $\mathfrak{A} \setminus \{0\} = \bigcup_{n\in\omega} \mathcal{A}_n$, where $\mathcal{A}_n$ is centered for each n, i.e. every finite subfamily of $\mathcal{A}_n$ has nonempty intersection. Show that $\mathfrak{A}$ is σ-centered if and only if $\mathrm{St}(\mathfrak{A})$ is separable.

Zad. 7 Let K be a compact space, $\mathcal{F}$ be a filter on ω and let (x_n) be a sequence in K. We say that $x \in K$ is a $\mathcal{F}$-limit of (x_n) ($\lim_{n\to\mathcal{F}} x_n = x$) if for every open $U \ni x$ the set $\{n\colon x_n \in U\} \in \mathcal{F}$.

- Show that if $\mathcal{F}$ is a filter of co-finite sets, then $\lim_{n\to\mathcal{F}} x_n = x$ if and only if $\lim_{n\to\infty} x_n = x$.
- What is $\lim_{n\in\mathcal{F}} x_n$ if $\mathcal{F}$ is a principal ultrafiler?
- Show that if $\mathcal{U}$ is an ultrafilter, then $\lim_{n\to\mathcal{U}} x_n$ exists for every (x_n).

Zad. 8 Show that every bounded function $f\colon \omega \to \mathbb{R}$ has a continuous extension to $f'\colon \beta\omega \to \mathbb{R}$. (Hint: use $\mathcal{F}$-limits.)

Zad. 9 Show that in $\beta\omega \setminus \omega$ every non-empty G_δ-set has a non-empty interior.

Zad. 10 Show that in $\beta\omega \setminus \omega$ every disjoint F_σ sets have disjoint closures.