Applications of infinitary combinatorics **2** 2018

Filters and ultrafilters.

Zad. 1 Show that every filter can be extended to an ultrafilter.

Zad. 2 We say that a family $\mathcal{A}$ generates a free filter if its closure under supersets and intersections is a free filter. Show that this is equivalent to the assertion that $\mathcal{A}$ has *strong finite intersection property* (sfip), i.e. every intersection of finitely many elements of $\mathcal{A}$ is infinite.

Zad. 3 Show that there are $2^{\mathfrak{c}}$ many ultrafilters on ω. Hint: consider an independent family $(A_\alpha)_{\alpha<\mathfrak{c}}$ of subsets of ω. For $f\colon \mathfrak{c} \to \{0,1\}$ show that there is an ultrafilter $\mathcal{F}_f$ such that $A_\alpha \in \mathcal{F}_f$ if and only if $f(\alpha) = 1$.

Zad. 4 Fix $n \in \omega$ and consider a family $\emptyset \notin \mathcal{C} \subseteq \mathcal{P}(\omega)$ satisfying the condition $(\star)_n$: "for every partition of ω in less than $n+1$ many sets the family $\mathcal{C}$ contains exactly one element of the partition". For which n the condition $(\star)_n$ is equivalent to "$\mathcal{C}$ is an ultrafilter"? Note: we do not assume here that $\mathcal{C}$ is a filter!

Zad. 5 If $\mathcal{F}$ is a filter on ω, then P is a pseudo-intersection of $\mathcal{F}$ if P is infinite and $P \subseteq^* F$ for every $F \in \mathcal{F}$.

- Show that every free filter generated by countably many elements does have a pseudo-intersection.
- Show that the density filter does not have a pseudo-intersection.

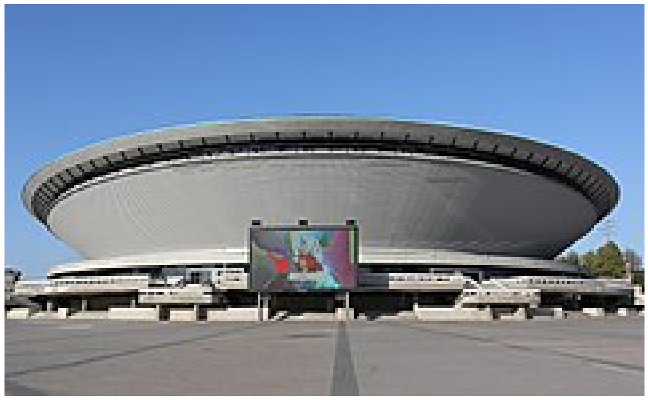
Zad. 6 A filter $\mathcal{F}$ is a P-filter if for every family $\{F_n\colon n < \omega\} \subseteq \mathcal{F}$ there is $A \in \mathcal{F}$ such that $A \subseteq^* F_n$ for every $n < \omega$.

- Show that under Continuum Hypothesis there is a P-ultrafiler.
- Show that the density filter, i.e.

  $$\{A \subseteq \omega\colon\ \lim \frac{A \cap \{0, \ldots, n-1\}}{n} = 1\}$$

  is a P-filter.
- Show that the density filter cannot be extended to a P-ultrafilter.

Zad. 7 (*) Is $\mathcal{P}(\omega)/\mathrm{Fin}$ isomorphic to $\mathcal{P}(\omega_1)/\mathrm{Fin}$?



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http://www.math.uni.wroc.pl/~pborod/dydaktyka