Applications of infinitary combinatorics **0** 2018

Well-ordered sets and transfinite induction.

Zad. 1 How big (in terms of cardinality) can be a well-ordered subset of $(\mathcal{P}(\mathbb{N}), \subseteq)$? How big can be a linearly ordered subset of $(\mathcal{P}(\mathbb{N}), \subseteq)$?

Zad. 2 Consider $(A_\alpha)_{\alpha\in\lambda}$, where λ is an ordinal number. Show that there is a family $(B_\alpha)_{\alpha\in\lambda}$ of pairwise disjoint sets such that for each α we have $B_\alpha \subseteq A_\alpha$ and

$$\bigcup_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda} B_\alpha.$$

Zad. 3 We say that $A \subseteq \mathbb{R}$ is $\mathfrak{c}$-*dense*, if it intersects every nonempty open interval on a set of cardinality $\mathfrak{c}$. Prove that $\mathbb{R}$ can be partitioned into $\mathfrak{c}$ many pairwise disjoint and $\mathfrak{c}$-dense sets.

Zad. 4 Prove that there is a set $A \subseteq \mathbb{R}^2$ which intersects every line in exactly two points.

Zad. 5 Show that $\mathbb{R}^3 \setminus \{\langle 0,0,0\rangle\}$ is a union of pairwise disjoint lines.

Zad. 6 Prove that every function $f\colon \mathbb{R} \to \mathbb{R}$ is a sum of two injective functions.

Zad. 7 Prove that every linear space has a base.

Zad. 8 Show that there is a family $(T_\alpha)_{\alpha<\omega_1}$ of subsets of $\mathbb{N}$ such that $T_\alpha \setminus T_\beta$ is infinite if and only if $\alpha < \beta$.

Zad. 9 Let $f\colon A \to \mathbb{R}$ be a continuous function ($A \subseteq \mathbb{R}$). Show that f can be extended to a continuous function $\hat{f}\colon B \to \mathbb{R}$, where $A \subseteq B$ and $B \in \Pi^0_2$.

Zad. 10 Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ which is not continuous on any set of size $\mathfrak{c}$. (Hint: use the previous exercise).

Zad. 11 Show that there is an ordinal number ϵ such that $\omega^\epsilon = \epsilon$.

Zad. 12 Show that every closed $A \subseteq \mathbb{R}$ is a union of a perfect set and a countable set.

Zad. 13 Construct a closed set of as high Cantor-Bendixson rank as you can.

Zad.* 14 Show that every Goodstein sequence converges to 0. (Check Wikipedia for what Goodstein sequence is.)

Zad. 15 Consider the following vending machine: if you drop 1 coin, it gives you 2 coins back. How to drop coins in such a way that after ω many steps you have no coins? Is it possible to do the same in ω_1 steps? Show that dropiing the coins randomly in ω steps we will have no coin at the end with probability 1.

A little cheating. Assume CH. In a bar two guys are playing darts. Professor Bufini offers to show them a trick: he will leave the room and then the players will throw one dart each. Professor says that after coming back he will be able to say, looking only at the dartboard, which dart was thrown as the first one. What's the trick? First, using CH, the professor enumerated all the points at dartboard by countable ordinals. Looking at the darts at dartboard the professor makes the following deduction: the dart which was thrown to a point with a lower number has to be the first one. Indeed, assume that the first dart hit a point labelled by α. Then the set of points labelled by numbers smaller than α is countable, and so it is of Lebesgue measure 0. Therefore, the second player with probability 1 hit a point of a higher number. Whats wrong with the professor Bufini's argument?

Pbn
http://www.math.uni.wroc.pl/~pborod/dydaktyka