Applications of infinitary combinatorics **10** 2019

Let $\mathcal{I}_{\frac{1}{n}}$ be the summable ideal, i.e.

$$\mathcal{I}_{\frac{1}{n}} = \{A \subseteq \omega \colon \sum_{i \in A} 1/i < \infty\}.$$

Let $\mathcal{Z}$ be the density ideal, i.e.

$$\mathcal{Z} = \{A \subseteq \omega \colon \lim \frac{|A \cap n|}{n} = 0.\}$$

Zad. 1 Show that the summable ideal and the ideal of density 0 sets are both dense. Show that they are P-ideals. Show that they are Borel. Which of them is F_σ?

Zad. 2 Show that $\mathcal{I}_{\frac{1}{n}} \leq_K \mathcal{Z}$.

Zad. 3 For an LSC submeasure φ define

$$\mathrm{Fin}(\varphi) = \{A \subseteq \omega \colon \varphi(A) < \infty\}.$$

$$\mathrm{Exh}(\varphi) = \{A \subseteq \omega \colon \lim_n \varphi(A \setminus n) = 0\}.$$

- Show that $\mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi)$.
- Show that $\mathrm{Fin}(\varphi)$ is an ideal. Show that it is F_σ.
- Give an example of a submeasure φ for which $\mathrm{Exh}(\varphi) \neq \mathrm{Fin}(\varphi)$.
- Give an example of a submeasure φ for which $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$.
- Show that $\mathcal{I}$ is dense if and only if $\lim_n \varphi(\{n\}) = 0$, where $\mathcal{I} = \mathrm{Exh}(\varphi)$.

Zad. 4 Show that

$$\mathcal{Z} = \{I \subseteq \omega \colon \limsup_n \frac{|[2^n, 2^{n+1}) \cap I|}{2^n} = 0\}.$$

Zad. 5 Show that $\mathrm{tr}(\mathcal{N})$ is neither isomorphic to the summable ideal, nor to the density 0 ideal.

Zad. 6 The ideal NWD is the ideal of nowhere dense subsets of $\mathbb{Q}$. Show that it is $F_{\sigma\delta}$. Is it a P-ideal?