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# On the Riesz Integral Representation of Additive Set-Valued Maps (II)

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Let $T$ be a compact topological space, and let $C_+(T)$ be the space of all non-negative continuous real-valued functions defined on $T$ endowed with the topology of uniform convergence. We prove the Riesz integral representation for continuous additive and positive set-valued maps defined on $C_+(T)$ with values in the space $cc(E)$ of all weakly compact convex non-empty subsets of a Banach space $E$. As an application we give a generalization of Dunford-Schwartz's result on the Riesz integral representation for any continuous set-valued map (not necessary positive).

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## 1. Introduction

In [4], the Riesz integral representation for continuous linear maps associated with additive multifunctions defined from an algebra $\mathfrak{A}$ of subsets of a non-empty set $T$ to the space $cfb(E)$ of all bounded closed convex non-empty subsets of a Banach space $E$ was developed. As an application theorems on representation were deduced for the multifunctions, for vector valued maps and for scalar valued maps. In this paper, $T$ is a compact topological space and $\mathfrak{B}$ is the $\sigma$-algebra of Borel subsets of $T$. We prove the Riesz integral representation for continuous additive and positive multifunctions defined from $C_+(T)$ to the space $cc(E)$ of all weakly compact convex non-empty subsets of $E$ : any continuous additive, positive and positively homogeneous multifunction $L$ from $C_+(T)$ to $cc(E)$ is of the form $L(f) = \int f\,dM$ for all $f \in C_+(T)$, where $M$ is a positive regular multimeasure from $\mathfrak{B}$ to $cc(E)$. The space $C_+(T)$ (resp. $cc(E)$) is endowed with the topology of uniform convergence (resp. the Hausdorff distance). This result is a generalization of Rupp's one (see [7], theorem 2) where the Banach space $E$ is finite dimensional. That was also partially known to Pallu de la Barrière (see

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[6], theorem 7-1, p. 3-26). We deduce from this result a representation theorem for any continuous additive multifunction (not necessary positive) defined from $C_+(T)$ to $cc(E)$.

## 2. Notations and definitions

The notations and definitions introduced in [4] are preserved here. Let $T$ be a compact topological space, let $C(T)$ be the space of all continuous real-valued functions defined on $T$ and let $C_+(T)$ be the subspace of $C(T)$ consisting of non-negative functions. The space $C(T)$ is endowed with the topology of uniform convergence. Measures are always countably additive set functions. Let $E$ be a Banach space, $E'$ its dual and $E''$ its bidual. $\sigma(E', E)$ and $\sigma(E'', E')$ are weak$^*$ topologies on $E'$ and $E''$, respectively and, $cc(E)$ is the family of all weakly compact convex non-empty subsets of $E$. Note that $cc(E)$ is a closed subset of the metric space $cfb(E)$ endowed with the Hausdorff distance $\delta$ (see [6]). $cc(E'', E')$ is the set of all $\sigma(E'', E')$-compact non-empty convex subsets of $E''$.

**Definition 2.1.** (1) Let $M$ be a multifunction from $\mathfrak{B}$ to $cc(E)$. We say that $M$ is a *multimeasure* if $M$ is additive and if

$$M\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} M(A_n)$$

for each sequence $(A_n)$ of pairwise disjoint elements of $\mathfrak{B}$; which amounts to saying that for all $y \in E'$, the map $\delta^*(y|M) : \mathfrak{B} \to \mathbb{R}$ $(A \mapsto \delta^*(y|M(A)))$ is a real valued measure (see [6], corollary, p. 2-25).

(2) Let $\mu : \mathfrak{B} \to \mathbb{R}$ be a positive measure. We say that $\mu$ is:
  (i) *inner regular* if for all $A \in \mathfrak{B}$ $\mu(A) = \sup\{\mu(K); K \text{ compact } K \subset A\}$.
  (ii) *outer regular* if for all $A \in \mathfrak{B}$ $\mu(A) = \inf\{\mu(O); O \subset T,\, O \text{ open } O \supset A\}$
  (iii) *regular* if it is inner and outer regular.

(3) A signed measure $\mu : \mathfrak{B} \to \mathbb{R}$ is *regular* if its total variation is regular.

(4) A multimeasure $M : \mathfrak{B} \to cc(E)$ is called *regular* if for each $y \in E'$ the measure $\delta^*(y|M)$ is regular.

Let $M\colon \mathfrak{B} \to cc(E)$ be a multimeasure and let $f$ be a nonnegative continuous function defined on $T$. Then the integral of $f$ with respect to $M$ is defined as in [4]. Since $cc(E)$ is closed in $cfb(E)$, it is a complete subspace of the space $(cfb(E), \delta)$. Note that if $f$ and $g$ are nonnegative continuous functions such that $f \le g$, then $\int f\,dM \subset \int g\,dM$. Moreover for all $y \in E'$ $\delta^*\left(y \,\middle|\, \int f\,dM\right) = \int f\,d\delta^*(y|M)$ and $\int f\,dM \in cc(E)$.

Let $L\colon C_+(T) \to cc(E)$ be an additive and positively homogeneous multifunction. We say that $L$ is bounded (resp. positive) if $\cup\{L(f)\colon f \in C_+(T),\, \|f\| \le 1\}$ is a bounded subset of $E$ (resp. $0 \in L(f)$ for all $f \in C_+(T)$). Note that a positive and positively homogeneous multifunction is bounded.

**Definition 2.2.** Let $F$ be a real vector space and let $s$ be a functional defined on $F$. We say that $s$ is *sublinear* if $s(x+y) \leq s(x) + s(y)$ and $s(\alpha x) = \alpha s(x)$ for all $x, y \in F$ and for all $\alpha \geq 0$.

**Definition 2.3.** Let $L\colon C_+(T) \to cc(E)$ be an additive, positively homogeneous and continuous multifunction. A *selection* of $L$ is a linear map $l\colon C(T) \to E$, which verifies $l(f) \in L(f)$ for all $f \in C_+(T)$.

**Definition 2.4.** Let $E$ and $F$ be two Banach spaces and let $l$ be a linear map from $E$ to $F$. Then $l$ is said to be *weakly compact* if it maps the closed unit ball of $E$ into a relatively weakly compact subset of $F$.

We denote by $\mathcal{M}^r(\mathfrak{B}, cc(E))$(resp. $\mathcal{M}^r(\mathfrak{B}, E)$) the set of all regular multimeasures (resp. vector measures) defined on $\mathfrak{B}$ with values in $cc(E)$ (resp. in $E$) and by $\mathcal{L}^c(C_+(T), cc(E))$ the set of all additive, positively homogeneous and continuous multifunctions from $C_+(T)$ to $cc(E)$.

If $E = \mathbb{R}$, then the space $\mathcal{M}^r(\mathfrak{B}, \mathbb{R})$ will be denoted by $\mathcal{M}^r(\mathfrak{B})$.

The following lemma is well-known (see [2, Theorem 5, p. 182]).

**Lemma 2.5.** *Let $E$ be a Banach space and $E'$ be its dual space endowed with the Mackey topology $\tau(E', E)$. Let $s$ be a sublinear functional defined on $E'$. Then $s$ is Mackey continuous if and only if there is $C \in cc(E)$ such that $s = \delta^*(\cdot|C)$.*

It follows that for each $f \in C_+(T)$ the map $y \mapsto \int f\, d\delta^*(y|M)$ from $E'$ to $\mathbb{R}$, is $\tau(E', E)$-continuous because $\int f\, dM \in cc(E)$.

**Theorem 2.6.** *Let $L \in \mathcal{L}^c(C_+(T), cc(E))$ be a positive multifunction. Then there exists a unique positive multimeasure $M \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ such that $L(f) = \int f\, dM$ for all $f \in C_+(T)$.*

*Conversely, for each positive multimeasure $M \in \mathcal{M}^r(\mathfrak{B}, cc(E))$, the multifunction $f \mapsto \int f\, dM$ from $C_+(T)$ to $cc(E)$ is a positive element of $\mathcal{L}^c(C_+(T), cc(E))$.*

**Proof.** Let $L$ be a positive element of $\mathcal{L}^c(C_+(T), cc(E))$ and let $y \in E'$. Then the functional $\delta^*(y|L)$ defined on $C_+(T)$ by $\delta^*(y|L)(f) := \delta^*(y|L(f))$ is additive, positively homogeneous and positive. Therefore $\delta^*(y|L)$ has a unique continuous linear extension on $C(T)$ denoted by $\delta^*(y|\overline{L})$. Simply, if $C(T) \ni f = f^+ - f^-$, then $\delta^*(y|\overline{L}(f)) := \delta^*(y|L(f^+)) - \delta^*(y|L(f^-))$.

Since $\delta^*(y|\overline{L})$ is linear, positive and continuous, by the Riesz representation theorem (cf. [1, Theorem IV.6.3, p. 265]) there exists a unique positive regular measure $\mu_y$ on $\mathfrak{B}$ such that $\delta^*(y|\overline{L}(f)) = \int f\, d\mu_y$ for all $f \in C(T)$.

Let $O$ be an open subset of $T$ and let $S_O$ be the functional defined on $E'$ by $S_O(y) = \mu_y(O)$. We have

$$\mu_y(O) = \sup\{\delta^*(y|L(f));\ f \in C_+(T),\ f \leq 1_O\}\,.$$

If $A$ is a member of $\mathfrak{B}$, we denote by $S_A$ the functional defined on $E'$ by $S_A(y) = \mu_y(A)$ for each $y \in E'$. Since $\mu_y$ is regular, we have

$$S_A(y) = \inf\{\mu_y(O); O \subset T,\ O \text{ open},\ O \supset A\}$$
$$= \sup\{\mu_y(K) : K \subset A,\ K \text{ compact}\}\,.$$

An easy consequence of the second representation is the sublinearity of $S_A$. A standard calculation proves the $\tau(E', E)$-continuity of $S_A$, for each $A \in \mathfrak{B}$. Then, by Lemma 2.5, there exists an element $C_A$ of $cc(E)$ such that $S_A(y) = \delta^*(y|C_A)$ for all $y \in E'$. Let $M\colon \mathfrak{B} \to cc(E)$ be the multifunction defined by $M(A) = C_A$. Then we have $\delta^*(y|M(A)) = S_A(y) = \mu_y(A)$ for all $A \in \mathfrak{B}$ and for all $y \in E'$. Therefore the map $\delta^*(y|M)\colon \mathfrak{B} \to \mathbb{R}$ is a positive regular measure. Hence $M \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ and

$$\delta^*(y|L(f)) = \int f\mu_y = \int f\delta^*(y|M) \text{ for all } f \in C_+(T),\ y \in E'\,. \tag{1}$$

Hence $L(f) = \int fM$ for all $f \in C_+(T)$.

Let us prove the uniqueness. Assume that there exist two multifunctions $M$ and $M'$ which verify (1). According to the inner regularity of the scalar measures $\delta^*(y|M)$ and $\delta^*(y|M')$ and the equality $\int f\delta^*(y|M) = \int f\delta^*(y|M')$ for all $f \in C_+(T), y \in E'$, we have $\delta^*(y|M(A)) = \delta^*(y|M'(A))$ for all $A \in \mathfrak{B}, y \in E'$ (due to the classical Riesz integral representation theorem). Since $M(A), M'(A) \in cc(E)$, we have $M(A) = M'(A)$ for all $A \in \mathfrak{B}$.

The second part follows from the properties of the integral with respect to the multimeasures and the inequality $\delta\left(\int fM, \int gM\right) \le \|f - g\|\|M\|(T)$ for all $f, g \in C_+(T)$. We recall that $\|f - g\| = \sup\{|f(t) - g(t)|;\ t \in T\}$ and $\|M\|(T) = \sup\{|\delta^*(y|M)|(T); y \in E', \|y\| \le 1\}$. □

The following corollary generalizes partly the well-know theorem of Dunford-Schwartz ([1, Theorem VI.7.2, p.492]).

**Corollary 2.7.** *Let $L \in \mathcal{L}^c(C_+(T), cc(E))$ be an arbitrary map. Then there is a unique multimeasure $M\colon \mathfrak{B} \to cc(E'', E')$, a positive multimeasure $M'\colon \mathfrak{B} \to cc(E)$ and a weak* measure $m'\colon \mathfrak{B} \to E''$ (that is $y \circ m'$ is a scalar measure for every $y \in E'$) such that:*

(i) $M = M' + m'$;

(ii) $\delta^*(y|M)$, $\delta^*(y|M')$ *and* $y \circ m'$ *are regular for each* $y \in E'$.

(iii) *for each* $f \in C_+(T)$ *the mappings:* $y \mapsto \int f\, d\delta^*(y|M)$, $y \mapsto \int f\, d\delta^*(y|M')$, $y \mapsto \int f\, dy \circ m'$ *are* $\tau(E', E)$*-continuous on* $E'$.

(iv) $\delta^*(y|L(f)) = \int f\, d\delta^*(y|M)$ *for each* $f \in C_+(T)$ *and* $y \in E'$.

*Conversely, if $M$ is a multifunction from $\mathfrak{B}$ to $cc(E'', E')$ which satisfies* (i)–(iii), *then there exists a multifunction $L \in \mathcal{L}^c(C_+(T), cc(E'', E'))$ such that $\delta^*(y|L) = \int f\, d\delta^*(y|M)$ for all $f \in C_+(T), y \in E'$.*

**Proof.** Let $L$ be a multifunction of $\mathcal{L}^c(C_+(T), cc(E))$ and let $l$ be a continuous, additive and positively homogenous selection of $L$ (see [6, Theorem 4.2, p. 3-14]). Let us put $L' = L - l$. Then $L' \in \mathcal{L}^c(C_+(T), cc(E))$ and is positive. By

Theorem 2.6, there exists a positive multimeasure $M' \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ such that $L'(f) = \int f\, dM'$ for all $f \in C_+(T)$. We obtain also the $\tau(E', E)$-continuity of the map $y \mapsto \int f\, d\delta^*(y|M')$.

Moreover, by the theorem of Dunford-Schwartz ([1, Theorem VI.7.2, p. 492]) there exists a unique set function $m'$ from $\mathfrak{B}$ to $E''$ such that

(1) $y \circ m' \in \mathcal{M}^r(\mathfrak{B})$ for each $y \in E'$;

(2) for each $f \in C_+(T)$, the mapping $y \mapsto \int f\, dy \circ m'$ of $E'$ into $\mathbb{R}$ is continuous for the topology $\sigma(E', E)$;

(3) $y \circ l(f) = \int f\, dy \circ m'$ for all $f \in C(T), y \in E'$.

Let us put $M(A) = M'(A) + m'(A)$, for all $A \in \mathfrak{B}$. We have then $M(A) \in cc(E'', E')$ and $\delta^*(y|M(A)) = \delta^*(y|M'(A)) + y \circ m'(A)$. Therefore $\delta^*(y|M) \in \mathcal{M}^r(\mathfrak{B})$.

If $f \in C_+(T)$, then the mapping $y \mapsto \int f\, dy \circ m'$ from $E'$ to $\mathbb{R}$ is continuous for the Mackey topology $\tau(E', E)$ because it is continuous for the weak* topology $\sigma(E', E)$. Therefore for each fixed $f \in C_+(T)$, also the mapping $y \mapsto \int f\, d\delta^*(y|M)$ from $E'$ to $\mathbb{R}$ is $\tau(E', E)$-continuous on $E'$.

Let $y \in E'$ and let $f \in C_+(T)$. We have

$$\begin{aligned}
&\delta^*(y|L(f)) = \delta^*(y|L'(f) + \{l(f)\}) \\
&= \delta^*(y|L'(f)) + y \circ m'(f) = \int f\, d\delta^*(y|M') + \int f\, dy \circ m' \\
&= \int f\, d\left(\delta^*(y|M') + y \circ m'\right) = \int f\, d\delta^*\left(y|M' + y \circ m'\right) = \int f\, d\delta^*(y|M)\,.
\end{aligned}$$

The uniqueness of $M$ can be proved in the same way as in Theorem 2.6. $M'$ and $m'$ are uniquely determined by $l$.

Conversely let $M\colon \mathfrak{B} \to cc(E'', E')$ be a multifunction which verifies (i), (ii) and (iii). Let $f \in C_+(T)$. The mapping $y \mapsto \int f\, d\delta^*(y|M')$ from $E'$ to $\mathbb{R}$ is sublinear.Then by (iii) and Lemma 2.5, there is $C_f \in cc(E)$ such that $\delta^*(y|C_f) = \int f\, d\delta^*(y|M')$.

Let $L'\colon C_+(T) \to cc(E)$ be the multifunction defined by $L'(f) = C_f$. Then $L'$ is additive and positively homogeneous. Moreover, if $\varepsilon > 0$, then

$$\begin{aligned}
&\sup\{|\delta^*(y|L(f))|; f \in C_+(T), \|f\| \le \varepsilon\} \\
&= \sup\left\{\left|\int f\, d\delta^*(y|M)\right|; f \in C_+(T), \|f\| \le \varepsilon\right\} \le \varepsilon|\delta^*(y|M)|(T)
\end{aligned}$$

(see [4]). This shows that $L'$ is continuous.

According to [1, Theorem VI.7.2, p. 492] the weak* measure $m'\colon \mathfrak{B} \to E''$ uniquely defines a linear operator $l\colon C(T) \to E''$ such that $y \circ l(f) = \int f\, dy \circ m'$ for every $f \in C(T)$. We define $L := L' + l$. $\square$

**Corollary 2.8.** *Let $L \in \mathcal{L}^c(C(T), cc(E))$. If $L$ has a weakly compact linear selection then there is a unique multimeasure $M \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ such that $L(f) = \int f\, dM$ for all $f \in C_+(T)$. Moreover, the selection uniquely determines a positive multimeasure $M' \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ and $m' \in \mathcal{M}^r(\mathfrak{B}, E)$ such that $M = M' + m'$.*

**Proof.** Let $l$ be a weakly compact linear selection of $L$. Then $L = L' + l$ where $L' = L - l$. Therefore, there exists $m' \in \mathcal{M}^r(\mathfrak{B}, E)$ such that $l(f) = \int f\, dm'$ for all $f \in C(T)$ (see [1, Theorem VI.7.3, p. 493]). In virtue of Theorem 2.6 there exists a positive multimeasure $M' \in \mathcal{M}^r(\mathfrak{B}, cc(E))$ which verifies $L'(f) = \int f\, dM'$ for all $f \in C_+(T)$. The multimeasure $M = M' + m'$ defined by $M(A) = M'(A) + m'(A)$ for all $A \in \mathfrak{B}$ satisfies the required conditions. □

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