CALCULUS

PROBLEMS LIST 3

11.10.2011

(1) Establish the natural domains of the following functions:

(a) $f(x) = \dfrac{\sqrt{x}}{\sin \pi x}$, (b) $f(x) = \sqrt{2 + x - x^2}$,
(c) $f(x) = \sqrt{3x - x^3}$, (d) $f(x) = \log(x^2 - 4)$,
(e) $f(x) = \log(1 - 2\cos x)$, (f) $f(x) = \sqrt{\sin(\sqrt{x})}$.

(2) Write as a formula $y = f(x)$ the composition of the following functions, and determine its natural domain:
(a) $t = 2^x,\ z = \sqrt[3]{t+1},\ y = z^2$, (b) $t = \sin x,\ z = \log t,\ y = \sqrt{1 + z^2}$.

(3) Sketch the graph of the function:
(a) $y = |x + 1| + |x - 1|$, (b) $y = |x - 3| - 2\,|x + 1| + 2\,|x| - x + 1$,
(c) $y = x^3 + 3x^2$, (d) $y = -x^3 + 2x - 2$,
(e) $y = 1 - \sin x$, (f) $y = 2\sin\left(x - \dfrac{\pi}{3}\right)$,
(g) $y = |\sin x|$, (h) $y = \dfrac{1}{\cos x}$.

(4) Find the inverse functions to:
(a) $y = 1 - 3x$, (b) $y = \dfrac{1}{1 - x},\ x \neq 1$,
(c) $y = x^2 - 2x,\ x \geq 1$, (d) $y = \sqrt[3]{x^2 + 1},\ x \geq 0$,

(5) Find the first 10 terms and the limit of the sequence $\{a_n\}$ given by the formula: $a_n = \dfrac{(-1)^n}{n^2}$.

(6) What are the values taken by the sequence: $a_n = \sin \dfrac{n\pi}{2}$?
And the sequence $a_n = \cos \dfrac{n\pi}{3} + i\,\sin \dfrac{n\pi}{3}$?

(7) The Fibonacci sequence is defined inductively in the following way: $F_1 = F_2 = 1$, and then $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, 3, \ldots$. Compute terms of this sequence numbered from 3 till 12. Prove, that for every natural number n the following inequality holds: $F_{n+2} \cdot F_n - F_{n+1}^2 = (-1)^{n+1}$.

(8) Using **only** the definition, prove the convergence of the following sequences, by finding their limits:
(a) $a_n = \dfrac{1}{n^2}$, (b) $a_n = \dfrac{(-1)^n}{n}$,
(c) $a_n = \left(\dfrac{2}{3}\right)^n$, (d) $a_n = \dfrac{n+2}{n-1},\ n \geq 2$,
(e) $a_n = \dfrac{1}{1 + \sqrt{n}}$, (f) $a_n = \dfrac{3n^3 - 2n^2 - 7n + 5}{4n^3 + n - 6}$.

(9) Prove that if x is a real number with the decimal expansion

$$\beta, \alpha_1\alpha_2 \cdots,$$

then the sequence given by the formula

$$a_n = \beta, \alpha_1 \cdots \alpha_n$$

is convergent to x (, is the decimal point and $\beta \in \mathbf{Z}$).

(10) Prove that the limit of the sum (difference, quotient) of convergent sequences is the sum (difference, quotient) of their limits. Of course, in the case of the quotient we assume that the sequence in the denominator had non-zero terms and its limit is different from zero.

(11) Check the monotonicity of the sequences:

(a) $a_n = n + \frac{1}{n}$, (b) $a_1 = 3,\ a_{n+1} = a_n^2 - 2$,

(c) $a_n = \sqrt[n]{n!}$, (d) $a_n = \sqrt[n]{2^n + 3^n}$

(e) $a_n = \frac{2^n}{n!}$, (f) $a_1 = 1,\ a_{n+1} = \frac{a_n}{1 + a_n}$.

(12) Find the limits (perhaps improper) of the sequences:

(a) $a_n = \frac{7n + (\sqrt[3]{n}\sqrt[6]{n})^5\sqrt{9n+1}}{11n^3 + 7n + 3}$, (b) $a_n = \sqrt{n^2 + n} - n$,

(c) $a_n = \frac{\sin n}{n}$, (d) $a_n = r^n,\ r > 1$,

(e) $a_n = \sqrt[n]{r},\ 0 < r < 1$, (f) $a_n = 2^n - \frac{1}{n}$,

(g) $a_n = \frac{\sqrt[3]{n^2 + n}}{n + 2}$, (h) $a_n = \frac{1 + 2 + 4 + \cdots + 2^n}{1 + 3 + 9 + \cdots + 3^n}$,

(i) $a_n = \frac{1 - 2 + 3 - 4 + 5 - 6 + \cdots - 2n}{\sqrt{n^2 + 2}}$, (j) $a_n = \frac{1 + 2 + \cdots + n}{n^2}$,

(k) $a_n = \frac{1 + 3 + 9 + \cdots + 3^n}{3^n}$, (l) $a_n = \sqrt{3^n + 2^n}\sqrt{3^n + 1}$,

(m) $a_n = \sqrt[(n^2)]{n}$, (n) $a_n = \sqrt[n]{n^2}$,

(o) $a_n = n(\sqrt{n^2 + 7} - n)$, (p) $a_n = \frac{n^2 + n + 1}{(n + \sin n)^2}$,

(q) $a_n = \frac{n^2 + 1}{n^3 + 1} + \frac{n^2 + 2}{n^3 + 2} + \frac{n^2 + 3}{n^3 + 3} + \cdots + \frac{n^2 + n}{n^3 + n}$,

(r) $a_n = \frac{1}{n^2} + \frac{1}{n^2 + 1} + \frac{1}{n^2 + 2} + \cdots + \frac{1}{(n+1)^2}$,

(s) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+7} - \sqrt{n}}$, (t) $a_n = r^n,\ -1 < r < 1$.

(13) Write out the formula for a sequence for which $a_1 = 1,\ a_2 = \frac{1}{2}$, and each consecutive term is the harmonic average of its neighbors:

$$\frac{1}{a_n} = \frac{1}{2}\left(\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}\right), \quad n \geq 2.$$

(14) Write out the formula for a sequence for which $a_1 = 1,\ a_2 = 2$, and each consecutive term is the geometric average of its neighbors:

$$a_n = \sqrt{a_{n-1}a_{n+1}}, \quad n \geq 2.$$