CALCULUS

PROBLEMS LIST 1

27.09.2011

(1) Express $0,123(45)$ as a usual fraction.
(2) Express $0,1(270)$ as a usual fraction.
(3) Show that the expansion

$$x = 0,1234567891011121314151617181920212223\ldots$$

built up of the consecutive natural numbers is not periodic.
Hint: Justify that the above expansion has places in which there are two consecutive zeros, three zeros, four zeros etc., i.e. it contains arbitrarily long "segments" consisting of zeros.
(4) Find the first three decimal digits after the decimal point of $\sqrt[3]{7}$.
(5) Show that numbers $\sqrt{24}$ and $\sqrt[5]{10}$ are both irrational.
(6) Prove that the set of integers is neither bounded from above nor bounded from below.
Hint: Use the Archimedean axiom.
(7) Show that no rational number is the least upper bound of the set of rational numbers x satisfying $x^3 < 10$.
Note: The question is about a rational number.
(8) Give an example of an x such that:
(a) $0 < x < 1$ and x is irrational,
(b) $\sqrt{5} < x < \sqrt{6}$ and x is rational,
(c) x^2 and x^3 are both irrational, but x^5 is rational,
(d) x^4 and x^6 are both rational, but x^5 is irrational,
(e) $(x+1)^2$ is irrational,
(f) x is irrational, but $x + \frac{1}{x}$ is rational.
(9) Using the definition find the supremum and the infimum of the open interval $(1, 2)$.
(10) Find the supremum and the infimum of the set

$$\left\{\frac{1}{n} + \frac{1}{k}; n, k \in \mathbf{N}\right\}.$$

(11) Find the supremum and the infimum of the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots\right\}$$

consisting of the reciprocals of the consecutive natural numbers.
(12) Find the supremum and the infimum of the set

$$A = \left\{x \in \mathbf{R} : x^2 < 2\right\}$$

(13) Prove that the number $\sqrt{3} + \sqrt{6}$ is irrational.
(14) Prove that the number $\sqrt[3]{5} + \sqrt[3]{6}$ is irrational.
(15) Without the aid of a calculator find the integral parts of numbers of the form $(\sqrt[3]{4})^n$ for $n = 1, 2, \ldots, 5$.

Hint: Write out the cubes of consecutive natural numbers, and consecutive powers of 4, and then compare.

(16) Prove that every open interval (a, b) contains an irrational number.

(17) Prove that arbitrary real numbers x, y satisfy the inequality

$$\big||x| - |y|\big| \leq |x - y|.$$

(18) Prove that for any real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

(19) Find the supremum and the infimum of the set

$$\{x + y : \ x, y > 0, \ [x] + [y] = 3\}.$$

(20) Show that

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}, \quad \min\{x, y\} = \frac{x + y - |x - y|}{2},$$

where $\max\{x, y\}$ denotes the larger of the numbers x and y, and $\min\{x, y\}$ the smaller of these numbers.

(21) Show that $|a - b - c| \geq |a| - |b| - |c|$.

(22) Let $x = 1,0234107...$, $y = 1,0235106...$ Is it true that

(a) $1,02 < x \leq 1,03$?

(b) $x + y > 2,04692$?

(c) $x < y$?

(23) Describe, on the real axis the sets

(a) $\{x : |x - 3| < 2\}$,

(b) $\{x : |x - 1| < |x + 1|\}$

(c) $\{x : |a + 1| < |x - a| < |x + 1|\}$.

(24) Solve the following equations and inequalities:

(a) $|x + 1| = |x - 1|$,

(b) $|1 - 2x| + |2x - 6| = x$,

(c) $|3x| + 2 \leq |x - 6|$,

(d) $|x^2 - 25| \leq 24$,

(e) $|x| + |x + 1| + |x + 2| = x^2 + 2x + \frac{29}{9}$,

(f) $|x + 10| = |2x + 1| + 3$.

(25) Is it true, that for every real number x we have the inequality:

(a) $x \leq |x|$,

(b) $-x \leq x$,

(c) $1 \leq |1 + x| + x$,

(d) $-1 \leq |-1 + x| + x$,

(e) $1 \leq |1 - x| + x$,

(f) $-1 \leq |-1 - x| + x$,

(g) $x \leq |x + 1| + 1$,

(h) $-x \leq |-x + 1| + 1$,

(i) $x \leq |x - 1| + 1$,

(j) $-x \leq |-x - 1| + 1$.

(26) Prove the following formula:

$$1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + 5 \cdot 3^4 + \cdots + n \cdot 3^{n-1} = \frac{2n - 1}{4} \cdot 3^n + \frac{1}{4}.$$

(27) Prove the following formula:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

(28) Prove the following formula:

$$1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + 5 \cdot 2^4 + \cdots + n \cdot 2^{n-1} = (n - 1) \cdot 2^n + 1.$$