Stein's unbiased risk estimator

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Stein's identity in one dimension (1)

$$X \sim N(\mu, \sigma^2), \quad f(x) : \mathbf{R} \to \mathbf{R}\,, E|f(X)| < \infty, \quad E|f'(X)| < \infty$$

Stein's identity:

$$E[(X - \mu)f(X)] = \sigma^2 E[f'(X)]$$

Proof:

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\phi'_{\mu,\sigma}(x) = -\frac{(x-\mu)}{\sigma^2}\phi_{\mu,\sigma}(x)$$

Stein's identity in one dimension (2)

$$\sigma^2 E[f'(X)] = \sigma^2 \int_{-\infty}^{\infty} f'(z)\phi_{\mu,\sigma}(z)dz$$

$$= \sigma^2 f(z)\phi_{\mu,\sigma}(z)|_{-\infty}^{\infty} - \sigma^2 \int_{-\infty}^{\infty} f(z)\phi'_{\mu,\sigma}(z)dz$$

$$= \int_{-\infty}^{\infty} f(z)(z-\mu)\phi_{\mu,\sigma}(z) = E[f(X)(X-\mu)]$$

Stein's identity in p dimensions

$$X \sim N(\mu, \sigma^2 I), \quad f(x) : \mathbf{R^p} \to \mathbf{R^p}\,, \quad \text{weakly differentiable}$$

Stein's identity:

$$E[(X-\mu)^T f(X)] = \sigma^2 \sum_{i=1}^{p} E\left(\frac{\partial f_i(X)}{\partial X_i}\right) = \sigma^2 E[div\ f(X)]$$

Stein's Unbiased Risk Estimator (SURE) (1)

$\hat{\mu} = X + g(X)$, $g(X)$ satisfies Stein's identity

$$SURE(\hat{\mu}) = p\sigma^2 + ||g(X)||^2 + 2\sigma^2 div\ g(X)$$

Stein's Unbiased Risk Estimator (SURE) (2)

Theorem:

$$E[SURE(\hat{\mu})] = E||\hat{\mu} - \mu||^2$$

Proof:

$$E||\hat{\mu} - \mu||^2 = E||X + g(X) - \mu||^2$$

$$= E||X - \mu||^2 + E||g(X)||^2 + 2E[(X - \mu)^T g(X)]$$

$$= p\sigma^2 + E||g(X)||^2 + 2\sigma^2 E[div\ g(X)]$$

$$= E[SURE(\hat{\mu})]$$

Application: James-Stein Estimator

$$\hat{\mu} = \left(1 - \frac{(p-2)\sigma^2}{||X||^2}\right) X$$

$g(X) = -\frac{(p-2)\sigma^2}{||X||^2} X$, satisfies Stein's identity when$p > 2$

$$||g(X)||^2 = \frac{(p-2)^2\sigma^4}{||X||^2}$$

$$\frac{\partial g_i(X)}{\partial X_i} = -\frac{(p-2)\sigma^2}{||X||^4}(||X||^2 - 2X_i^2)$$

$$div\ g(X) = -\frac{(p-2)\sigma^2}{||X||^2}(p-2) = -\frac{(p-2)^2\sigma^2}{||X||^2}$$

$$||g(X)||^2 + 2\sigma^2 div\ g(X) = \frac{(p-2)^2\sigma^4}{||X||^2} - 2\frac{(p-2)^2\sigma^4}{||X||^2} = -2\frac{(p-2)^2\sigma^4}{||X||^2}$$