Mapping class groups and cohomology of Homeo(M, vol)

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joint work with M. Brandenbursky

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- $\mathrm{H}_b^\bullet(G)$ is the bounded cohomology of a discrete group
- Aim: $\Gamma_b^{\mathcal{M}}\colon \mathrm{H}_b^n(\mathcal{M}(M,*)) \to \mathrm{H}_b^n(\mathrm{Homeo}(M, vol))$

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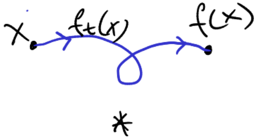

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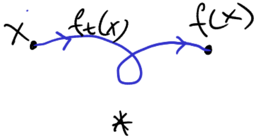

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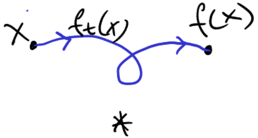

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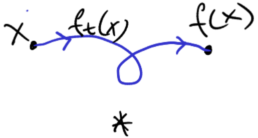

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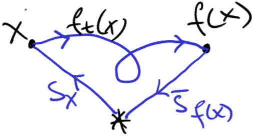

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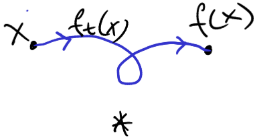

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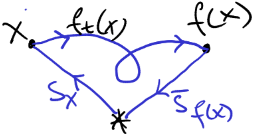
$\gamma(f, x) =$  $\in \pi_1(M, *)$

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By: $\Gamma_b(c)(f_0,\ldots,f_n) = \int_M c\big(\gamma(f_0,x),\ldots,\gamma(f_n,x)\big)dx$

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Martin Nitsche: Γ_b does not depend on the choice of F.

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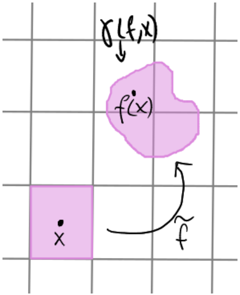

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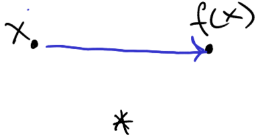

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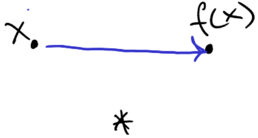

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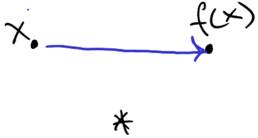

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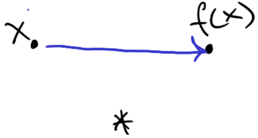

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$$\mathcal{M}(M,*) \circlearrowright \hat{M} \circlearrowleft \mathrm{Homeo}(M,vol)$$
$$\hat{M} \longrightarrow M$$

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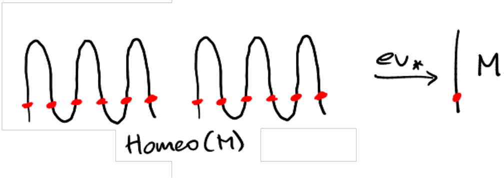

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In particular we can lift elements of $\text{Homeo}_0(M, vol)$ without selecting an isotopy.

Summary

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Theorem

$\Gamma_b^{\mathcal{M}}(e_b^{\mathcal{M}}) \in \mathrm{H}^2_b(\mathrm{Homeo}_+(S, area))$ *has a positive norm.*