BERNSTEIN SETS AND κ-COVERINGS

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Abstract. In this paper we study a notion of a κ-covering in connection with Bernstein sets and other types of nonmeasurability. Our results correspond to those obtained by Muthuvel in [7] and Nowik in [8]. We consider also other types of coverings.

1. Definitions and notation

In 1993 Carlson in his paper [3] introduced a notion of κ-coverings and used it for investigating whether some ideals are or are not κ-translatable. Later on κ-coverings were studied by other authors, e.g. Muthuvel (cf. [7]) and Nowik (cf. [8], [9]). In this paper we present new results on κ-coverings in connection with Bernstein sets. We also introduce two natural generalizations of the notion of κ-coverings, namely κ-S-coverings and κ-I-coverings.

We use standard set-theoretical notation and terminology from [1]. Recall that the cardinality of the set of all real numbers $\mathbb{R}$ is denoted by $\mathfrak{c}$. The cardinality of a set A is denoted by $|A|$. If κ is a cardinal number then

$$[A]^{\kappa} = \{B \subseteq A : \ |B| = \kappa\};$$
$$[A]^{<\kappa} = \{B \subseteq A : \ |B| < \kappa\}.$$

The cofinality of κ is denoted by $\mathrm{cf}(\kappa)$. The power set of a set A is denoted by $\mathcal{P}(A)$.

For a given uncountable Abelian Polish group $(X, +)$, the family of all uncountable perfect subsets of X is denoted by $\mathrm{Perf}(X)$ and the family of all Borel subsets of X is denoted by $\mathrm{Borel}(X)$. We say that a set $B \subseteq X$ is *a Bernstein set* if for every uncountable set $Z \in \mathrm{Borel}(X)$ both sets $Z \cap B$ and $Z \setminus B$ are nonempty.

In this paper $\mathcal{I}$ stands for a σ-ideal of subsets of a given uncountable Abelian Polish group $(X, +)$. We will always assume that $\mathcal{I}$ is proper and group invariant, contains singletons and has a Borel base (i.e. for every set $A \in \mathcal{I}$ we can find a Borel set $B \in \mathcal{I}$ such that $A \subseteq B$). We will use three cardinal characteristics of an ideal $\mathcal{I}$: the additivity number $\mathrm{add}(\mathcal{I})$, the covering number $\mathrm{cov}(\mathcal{I})$ and the uniformity number $\mathrm{non}(\mathcal{I})$, defined as follows:

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| \, : \, A \subseteq \mathcal{I} \, \wedge \, \bigcup A \notin \mathcal{I}\};$$
$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| \, : \, A \subseteq \mathcal{I} \, \wedge \, \bigcup A = X\};$$
$$\mathrm{non}(\mathcal{I}) = \min\{|A| \, : \, A \subseteq X \, \wedge \, A \notin \mathcal{I}\}.$$

Let us recall the notion investigated for instance in [4].

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Definition 1. Let $N \subseteq X$. We say that the set N is *completely $\mathcal{I}$-nonmeasurable* if

$$(\forall A \in \mathrm{Borel}(X) \setminus \mathcal{I})(A \cap N \notin \mathcal{I} \wedge A \cap (X \setminus N) \notin \mathcal{I}).$$

In particular, for the σ-ideal of Lebesgue null sets $\mathcal{N} \subseteq \mathcal{P}(\mathbb{R})$ we have that a set $N \subseteq \mathbb{R}$ is completely $\mathcal{N}$-nonmeasurable if and only if the inner measure of N and the inner measure of the complement of N are zero. One can observe that if $\mathcal{I}$ is a σ-ideal of our interest (i.e. having properties mentioned above) then every Bernstein set is completely $\mathcal{I}$-nonmeasurable. Hence the notion of a completely $\mathcal{I}$-nonmeasurable set generalizes the notion of a Bernstein set.

While constructing completely $\mathcal{I}$-nonmeasurable sets having interesting covering properties we will concentrate on σ-ideals including all unit spheres. Let us observe that classical σ-ideals such as the σ-ideal of null sets $\mathcal{N}$ and the σ-ideal of meager sets $\mathcal{M}$ have this property.

The following notion of a tiny set is very useful in recursive constructions of completely $\mathcal{I}$-nonmeasurable sets.

Definition 2. Let us fix a family $\mathcal{A} \subseteq \mathcal{I}$. We say that a perfect set $P \in \mathrm{Perf}(X)$ is *a tiny set with respect to $\mathcal{A}$* if

(1) $(\forall t \in X)(\forall A \in \mathcal{A})\, |(P + t) \cap A| \leq \omega$,
(2) $(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\exists t \in X)\ |(P + t) \cap B| = \mathfrak{c}$.

In [10] Rałowski proved the following useful lemma.

Lemma 1.1. *Let $\mathcal{A} \subseteq \mathcal{I}$. If there exists a perfect set $P \in \mathrm{Perf}(X)$, which is tiny with respect to $\mathcal{A}$ then*

$$\min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A} \wedge (\exists B \in \mathrm{Borel}(X) \setminus \mathcal{I})(B \subseteq \bigcup \mathcal{B})\} = \mathfrak{c}.$$

Definition 3. We say that the σ-ideal $\mathcal{I}$ has the *Steinhaus property* if for every set $A \in \mathcal{P}(X) \setminus \mathcal{I}$ and $B \in \mathrm{Borel}(X) \setminus \mathcal{I}$ the set $A - B = \{a - b : a \in A \wedge b \in B\}$ contains a nonempty open set.

It is known that the σ-ideal of null sets and the σ-ideal of meager sets have the Steinhaus property (even in more general context – cf. [2], [6]).

Let observe that the following fact holds.

Fact 1.2. *Let $Q \subseteq X$ be any dense countable subgroup of X. If the σ-ideal $\mathcal{I}$ has the Steinhaus property then for any set $B \in \mathrm{Borel}(X) \setminus \mathcal{I}$ we have $(B + Q)^c \in \mathcal{I}$.*

Proof. Let us fix $B \in \mathrm{Borel}(X) \setminus \mathcal{I}$ and let $B^* = B + Q$. Suppose that $(B^*)^c \notin \mathcal{I}$. Then by the Steinhaus property there exists a nonempty open set $U \subseteq X$ such that $U \subseteq (B^*)^c - B^*$. Hence there exist some $q \in Q$ and $b \in B^*$ such that $q + b \in (B^*)^c$. Since $Q + B^* = B^*$, we get $q + b \in B^* \cap (B^*)^c$ which is a contradiction. □

Now we will focus our attention on σ-ideals $\mathcal{N}$ and $\mathcal{M}$. The next lemma is probably folklore, but for the reader's convenience we present its proof.

Lemma 1.3. *Let $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$. Then*

$$(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\forall P \in \mathrm{Perf}(X))(\exists t \in X)\ |(t + P) \cap B| = \mathfrak{c}.$$

Proof. (Cichoń) Firstly, let us assume that $\mathrm{cov}(\mathcal{I}) > \omega_1$ and choose any subset $T \in [P]^{\omega_1}$ of a perfect set P. Let $B^* = B + Q$, where Q is a dense countable subgroup of X. From Fact 1.2 we deduce that $\bigcup_{t \in T}(t + B^*)^c \neq X$. Let us fix

$y \in \bigcap_{t\in T}(t + B^*)$. Then $T \subseteq -y + B^*$. Thus there exist $x \in X$ and $S \in [T]^{\omega_1}$ such that $S \subseteq x + B$. But $S \subseteq P$ and P is perfect, so $|(x + B) \cap P| = \mathfrak{c}$.

Now let V be a model of ZFC. There exists a generic extension $V[G]$ fulfilling condition $MA + \mathfrak{c} \neq \omega_1$. Consequently, $V[G] \models \mathrm{cov}(\mathcal{I}) > \omega_1$. But the following formula

$$(\forall P \in \mathrm{Perf}(X))(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\exists x \in X)\ |(x + P) \cap B| = \mathfrak{c}$$

is Π^1_3. So it holds also in the ground model V because by Shoenfield's absoluteness theorem (cf. [12]) Π^1_3 formulas are downward absolute. □

Remark 1. Another proof for the measure case was given by Ryll-Nardzewski. His proof was based on convolution measures. Yet another proof is due to Morayne, where density points of measure are used.

Remark 2. Let us observe that Lemma 1.3 remains true for any σ-ideal $\mathcal{I}$ having the Steinhaus property such that it is consistent that $\mathrm{cov}(\mathcal{I}) > \omega_1$ and Borel codes for sets from the ideal $\mathcal{I}$ are absolute between transitive models of ZFC.

Question 1. *Is there any nontrivial example of a σ-ideal, other than $\mathcal{M}$ and $\mathcal{N}$, fulfilling conditions mentioned in Remark 2?*

Lemma 1.3 gives us a simpler characterization of a tiny set in case $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$.

Corollary 1.4. *If $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$ then a perfect set P is a tiny set with respect to a family $\mathcal{A} \subseteq \mathcal{I}$ if*

$$(\forall t \in X)(\forall A \in \mathcal{A})\, |(P + t) \cap A| \leq \omega.$$

Let us notice this characterization is not true in general (as pointed by the referee):

Example 1.5 (given by the referee)**.** *Assume that the cofinality of the σ-ideal of meager subsets of $\mathbb{R}$ is ω_1 and $\mathfrak{c} > \omega_1$. Let $(A_\alpha : \alpha < \omega_1)$ be a cofinal tower, consisting of meager sets in $\mathbb{R}$. Let $X = \mathbb{R} \times \mathbb{R}$ and let $\mathcal{I}$ be the σ-ideal of subsets of X with meager projections on the first coordinate. Let $\mathcal{A} = \{A_\alpha \times \{0\} : \alpha < \omega_1\}$, $P = \{0\} \times \mathbb{R}$ and $B = \mathbb{R} \times \{0\}$. Then P is tiny with respect to $\mathcal{A}$ as $|P \cap (A_\alpha \times \{0\})| \leq 1$ for each $\alpha < \omega_1$. However, $B \in \mathrm{Borel}(X) \setminus \mathcal{I}$, $|B \cap P| = 1$ and $B \subseteq \bigcup \mathcal{A}$, so the conclusion of Lemma 1.1 fails.*

In our applications we will concentrate on families of unit spheres in $\mathbb{R}^n$.

Lemma 1.6. *Let $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$. Let $\mathcal{D}$ be a family of unit spheres of size less than continuum and let $B \in \mathrm{Borel}(\mathbb{R}^n) \setminus \mathcal{I}$. Then*

$$\left|B \setminus \bigcup \mathcal{D}\right| = \mathfrak{c}.$$

Proof. Observe that every line is a tiny set with respect to the family of all unit spheres. So according to Lemma 1.1 and Corollary 1.4 the set B cannot be covered by $\bigcup \mathcal{D}$. Hence $|B \setminus \bigcup \mathcal{D}| = \mathfrak{c}$. □

Lemma 1.6 remains true for every σ-ideal mentioned in Remark 2.

2. Coverings on the real line

In [3] Carlson introduced the following definition.

Definition 4. We say that the set $A \subseteq \mathbb{R}$ is *a κ-covering* if for every set $B \subseteq \mathbb{R}$ of cardinality κ there exists a real number $x \in \mathbb{R}$ such that $B + x \subseteq A$.

Analogously, a set $A \subseteq \mathbb{R}$ is *a $<\kappa$-covering* if every set $B \subseteq \mathbb{R}$ of cardinality less then κ can be translated into it (cf. [7]). Of course, these definitions are reasonable also for other uncountable Abelian Polish groups.

Nowik in his papers studied partitions of the Cantor space 2^ω into regular (Borel) ω-coverings. He constructed such a partition of size continuum ([8]) and a partition into two sets, one F_σ, one G_δ, having some special property. We present analogous and even stronger results concerning irregular (Bernstein) sets.

First we prove that we can find a partition of the real line into two Bernstein sets having no covering properties.

Theorem 2.1. *There exists a partition of the real line $\mathbb{R}$ into two sets A, B such that each of them is a Bernstein set and none of them is a 2-covering.*

Proof. Let $\mathrm{Perf}(\mathbb{R}) = \{P_\alpha : \alpha < \mathfrak{c}\}$ and $\mathbb{R} = \{r_\alpha : \alpha < \mathfrak{c}\}$ be fixed enumerations of all perfect subsets of the real line and of the reals, respectively. By transfinite induction we build two increasing sequences $(A_\alpha)_{\alpha<\mathfrak{c}}$, $(B_\alpha)_{\alpha<\mathfrak{c}}$ of subsets of $\mathbb{R}$ such that for every $\alpha < \mathfrak{c}$ the following conditions are satisfied:

(1) $|A_\alpha| = |B_\alpha| = |\alpha| \cdot \omega$;
(2) $r_\alpha \in A_\alpha \cup B_\alpha$;
(3) $A_\alpha \cap P_\alpha \neq \emptyset$, $B_\alpha \cap P_\alpha \neq \emptyset$;
(4) $A_\alpha \cap B_\alpha = \emptyset$.

Moreover, to ensure that A_α and B_α are not 2-coverings we want them to satisfy two more conditions:

(5) $(\forall x \in A_\alpha)(\{x-1,\ x+1\} \subseteq B_\alpha)$;
(6) $(\forall x \in B_\alpha)(\{x-1,\ x+1\} \subseteq A_\alpha)$.

Now, the set $\{0,1\}$ cannot be translated neither into A_α nor into B_α.

We are able to fulfill all these conditions because being at the αth step of our construction we know that $|\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)| < \mathfrak{c}$ and for every $\beta < \alpha$ we have $(A_\beta \cup B_\beta) + \mathbb{Z} = A_\beta \cup B_\beta$.

Finally, we put $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$ and $B = \bigcup_{\alpha<\mathfrak{c}} B_\alpha$. These sets are Bernstein sets because of (3), form a partition of $\mathbb{R}$ because of (2) and (4) and are not 2-coverings as neither are sets A_α and B_α. □

The next theorem is in contrast with the previous one.

Theorem 2.2. *There is a partition $\{B_\xi : \xi < \mathfrak{c}\}$ of the real line into Bernstein sets such that for every $\xi < \mathfrak{c}$ the set B_ξ is a $<cf(\mathfrak{c})$-covering.*

Proof. Let $\kappa = cf(\mathfrak{c})$ and let $(c_\alpha)_{\alpha<\kappa}$ be a cofinal increasing sequence of elements of $\mathfrak{c}$. Let us fix an increasing sequence $(R_\alpha)_{\alpha<\kappa}$ of subsets of $\mathbb{R}$ and a sequence $(\mathcal{P}_\alpha)_{\alpha<\kappa}$ of families of perfect subsets of $\mathbb{R}$ such that

$$\mathbb{R} = \bigcup_{\alpha<\kappa} R_\alpha, \quad \mathrm{Perf}(\mathbb{R}) = \bigcup_{\alpha<\kappa} \mathcal{P}_\alpha$$

and $|R_\alpha| = |\mathcal{P}_\alpha| = |c_\alpha|$.

By transfinite induction we build a sequence of families $(\{B^\alpha_\xi : \xi < c_\alpha\})_{\alpha<\kappa}$ satisfying the following conditions:

(1) for every $\alpha < \kappa$ and for every $\xi < c_\alpha$ we have $|B^\alpha_\xi| = |c_\alpha|$;
(2) for every $\alpha < \kappa$ sets from the family $\{B^\alpha_\xi : \xi < c_\alpha\}$ are pairwise disjoint;
(3) for every $\xi < \mathfrak{c}$ and every $\alpha_1 < \alpha_2 < \kappa$ such that $\xi < c_{\alpha_1}$ we have $B^{\alpha_1}_\xi \subseteq B^{\alpha_2}_\xi$;
(4) for every $\alpha < \kappa$ the intersection $B^\alpha_\xi \cap P$ is nonempty for every $\xi < c_\alpha$ and every perfect set P from the family $\mathcal{P}_\alpha$;
(5) for every $\alpha < \kappa$ and every $\xi < c_\alpha$ there exists $x \in \mathbb{R}$ such that $x + R_\alpha \subseteq B^\alpha_\xi$.

We obtain such a sequence as follows. Assume that we are at the αth step of the construction, so we have already built families $\{B^\beta_\xi : \xi < c_\beta\}$ for $\beta < \alpha$. One can observe that the cardinality of the union of all sets B^β_ξ constructed so far (let us denote this sum by S) is small:

$$|S| = \left| \bigcup_{\beta<\alpha} \bigcup_{\xi<c_\beta} B^\beta_\xi \right| \leq |c_\alpha| \cdot |c_\alpha| \cdot |\alpha| = |c_\alpha| < \mathfrak{c}.$$

For every $\xi < c_\alpha$ let us put

$$B^{<\alpha}_\xi = \bigcup_{\beta<\alpha} B^\beta_\xi$$

(the set $B^{<\alpha}_\xi$ is empty for $\bigcup_{\beta<\alpha} c_\beta \leq \xi < c_\alpha$). Let us notice that there are at most c_α many real numbers x such that $(x + R_\alpha) \cap S \neq \emptyset$. Hence we can recursively enlarge every set $B^{<\alpha}_\xi$ adding to it a set $x_\xi + R_\alpha$ for some $x_\xi \in \mathbb{R}$ and keeping all enlarged sets pairwise disjoint – it is enough to fulfill (5). To fulfill (4) we have to enlarge our sets once more adding recursively to each of them one point from every set $P \in \mathcal{P}_\alpha$. Again, we can do this without losing disjointness. As a result we obtain a family $\{B^\alpha_\xi : \xi < c_\alpha\}$ which fulfills conditions (2)–(5). But the condition (1) is also fulfilled because constructing every set B^α_ξ we have added $|c_\alpha|$ many new points.

Finally, we put

$$B_\xi = \bigcup_{\alpha<\kappa} B^\alpha_\xi$$

(assuming that $B^\alpha_\xi = \emptyset$ for $\alpha < \min\{\eta : \xi < c_\eta\}$).

Thanks to (2) the family $\{B_\xi : \xi < \mathfrak{c}\}$ consists of pairwise disjoint sets and without problems we can extend them to get a partition of $\mathbb{R}$. By (4) every set B_ξ is a Bernstein set. Moreover, the condition (5) is enough to ensure that every set B_ξ is a $<\kappa$-covering. It is because every subset of the real line of cardinality smaller than κ is a subset of one of the R_α's. □

On the other hand, as the only $\mathfrak{c}$-covering subset of the real line is the set $\mathbb{R}$ itself, we have the following fact.

Fact 2.3. *Assume CH. Then there is no Bernstein set which is an ω_1-covering.*

Now, one can pose the following question.

Question 2. *Assume $\mathfrak{c} > \omega_1 = \mathrm{cf}(\mathfrak{c})$. Is it true that there exists an ω_1-covering which is a Bernstein set?*

It is worth mentioning that in the proof of Theorem 2.2 we have succeeded in constructing relevant ω_1-coverings because we have been able to cover every set of size ω_1 by a set of size smaller then continuum, taken from the fixed family of size at most continuum. Let us notice that it is not possible to answer Question 2 using the similar method as in the proof of Theorem 2.2 since we have the following observation which is a special case of the fact that if λ is singular then $\mathrm{cov}(\lambda, \lambda, \mathrm{cf}(\lambda)^+, 2) > \lambda$.

Fact 2.4 (see [11]). *Assume that $\mathfrak{c} = \omega_{\omega_1}$. Then there is no family $\mathcal{B} \subseteq [\mathbb{R}]^{<\mathfrak{c}}$ of size continuum such that every subset of $\mathbb{R}$ of size ω_1 is covered by some set from the family $\mathcal{B}$.*

If we deal with completely $\mathcal{I}$-nonmeasurable sets instead of Bernstein sets then we can construct even a $<\mathfrak{c}$-covering on condition the σ-ideal $\mathcal{I}$ has the Steinhaus property and its uniformity is not too big.

Proposition 2.5. *Assume that $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is a σ-ideal having the Steinhaus property and such that* $\mathrm{non}(\mathcal{I}) < \mathfrak{c}$. *Then there exists a $<\mathfrak{c}$-covering which is completely $\mathcal{I}$-nonmeasurable.*

Proof. Let us fix a set $N \notin \mathcal{I}$ such that $|N| = \mathrm{non}(\mathcal{I})$ and put $C = (N + \mathbb{Q})^c$. Suppose now that $B \in \mathrm{Borel}(\mathbb{R}) \setminus \mathcal{I}$. Then from the Steinhaus property of $\mathcal{I}$ we obtain that there exists a rational $q \in \mathbb{Q}$ such that $q \in C^c - B$. Hence $B \cap C^c \neq \emptyset$. As $|C^c| < \mathfrak{c}$ we have also $B \cap C \neq \emptyset$, so the set C is completely $\mathcal{I}$-nonmeasurable.

Moreover, the set C is a $<\mathfrak{c}$-covering. Indeed, suppose that there exists a set $A \in [\mathbb{R}]^{<\mathfrak{c}}$ such that for every $x \in \mathbb{R}$ we obtain $(A + x) \cap C^c \neq \emptyset$. For every $x \in \mathbb{R}$ let us fix $a_x \in A$ such that $a_x + x \in C^c$. Then there exists $c \in C^c$ such that $|\{x \in \mathbb{R} : a_x + x = c\}| > |A|$. But all reals $c - x = a_x \in A$ are different and we have got a contradiction. □

3. S-coverings

We can interpret κ-coverings in terms of coloring sets. Namely, we can treat a κ-covering as set which can color every set of size κ monochromatically. From this point of view we may ask about a family of sets which can color every set of size κ in such a way that different points in the given set have different colors. This leads us to the following definition.

Definition 5. A family $\mathcal{A}$ of pairwise disjoint subsets of the real line is called a *κ-S-covering* if $|\mathcal{A}| = \kappa$ and

$$(\forall F \in [\mathbb{R}]^\kappa)(\exists t \in \mathbb{R})\Big(F + t \subseteq \bigcup \mathcal{A} \ \wedge \ (\forall A \in \mathcal{A})|(F + t) \cap A| = 1\Big).$$

This definition is reasonable also for other uncountable Abelian Polish groups.

First we prove a relation between 2-S-coverings and 2-coverings.

Theorem 3.1. *Assume that $\{A_0, A_1\}$ is a partition of the real line and a 2-S-covering. Then at least one of the sets A_0, A_1 is a 2-covering.*

Proof. Assume that none of the sets A_0, A_1 is a 2-covering. It means that there are positive reals a, b such that for every $x, y \in A_0$ we have $x - y \neq a$ and for every $x, y \in A_1$ we have $x - y \neq b$. We will show that the set $\{0, a + b\}$ cannot be S-covered by $\{A_0, A_1\}$.

Indeed, let us fix any $x \in A_0$. Then $x + a \in A_1$ and, consequently, $x + a + b \in A_0$. Analogously, if $x \in A_1$ then $x + b + a \in A_1$, which ends the proof. □

Now we focus our attention on constructing κ-S-coverings which consist of Bernstein sets or completely $\mathcal{I}$-nonmeasurable sets and such that none of their elements is a κ-covering (which is opposite to the situation from Theorem 3.1).

Theorem 3.2. *Let κ be a cardinal number such that $2 < \kappa < \mathfrak{c}$. If $2^\kappa \leq \mathfrak{c}$ then there exists a partition $\{B_\xi : \xi < \kappa\}$ of the real line such that*

(1) *$(\forall \xi < \kappa)\, B_\xi$ is a Bernstein set,*
(2) *$(\forall \xi < \kappa)\, B_\xi$ is not a 2-covering,*
(3) *$\{B_\xi : \xi < \kappa\}$ is a κ-S-covering.*

Proof. Let $\mathrm{Perf}(\mathbb{R}) = \{P_\alpha : \alpha < \mathfrak{c}\}$ and $\mathbb{R} = \{r_\alpha : \alpha < \mathfrak{c}\}$ be fixed enumerations of all perfect subsets of the real line and of the reals, respectively. Let us also enumerate the set $[\mathbb{R}]^\kappa = \{F_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction we build a sequence $(\{A^\alpha_\xi : \xi < \kappa\})_{\alpha<\mathfrak{c}}$ of families of subsets of $\mathbb{R}$ of size less than continuum such that for every $\alpha < \mathfrak{c}$ the following conditions are fulfilled:

(1) for every different $\xi_1, \xi_2 < \kappa$ the sets $A^\alpha_{\xi_1}$ and $A^\alpha_{\xi_2}$ are disjoint;
(2) for every $\xi < \kappa$ the intersection $A^\alpha_\xi \cap P_\alpha$ is nonempty;
(3) there exists $t_\alpha \in \mathbb{R}$ such that $t_\alpha + F_\alpha \subseteq \bigcup_{\xi<\kappa} A^\alpha_\xi$ and for every $\xi < \kappa$ we have $|(t_\alpha + F_\alpha) \cap A^\alpha_\xi| = 1$;
(4) there exists $\xi < \kappa$ such that $r_\alpha \in A^\alpha_\xi$;
(5) for every $\xi < \kappa$ and every $\beta < \alpha$ we have $A^\beta_\xi \subseteq A^\alpha_\xi$;
(6) for every $\xi < \kappa$ and every $x, y \in A^\alpha_\xi$ we have $|x - y| \neq 1$;
(7) for every $\xi < \kappa$ we have $|A^\alpha_\xi| \leq |\alpha| \cdot \omega$.

Suppose that we have already constructed the sequence $(\{A^\beta_\xi : \xi < \kappa\})_{\beta<\alpha}$ for some $\alpha < \mathfrak{c}$. Let $A_\xi = \bigcup_{\beta<\alpha} A^\beta_\xi$ and $A = \bigcup_{\xi<\kappa} A_\xi$. We can observe that there are not many "bad translations" of the set F_α, namely the set

$$T = \{t \in \mathbb{R} : (\exists x \in F_\alpha)(\exists a \in A)\, |t + x - a| = 1 \,\vee\, t + x = a\}$$

has the cardinality less then $\mathfrak{c}$. Thus we can choose a real $t_\alpha \notin T$. Next we choose a subset $Y \subseteq P_\alpha$ of size κ such that

$$(Y + \{0, 1, -1\}) \cap ((t_\alpha + F_\alpha) \cup A) = \emptyset.$$

Let $\{a_\xi : \xi < \kappa\}$ and $\{b_\xi : \xi < \kappa\}$ be enumerations of sets $t_\alpha + F_\alpha$ and Y, respectively, and let $\hat{A}^\alpha_\xi = A_\xi \cup \{a_\xi, b_\xi\}$ for $\xi < \kappa$. Finally, if $r_\alpha \notin Y \cup (t_\alpha + F_\alpha) \cup A$ then we fix $\xi_0 < \kappa$ such that $\hat{A}^\alpha_{\xi_0} \cap \{r_\alpha - 1, r_\alpha + 1\} = \emptyset$ and put $A^\alpha_{\xi_0} = \hat{A}^\alpha_{\xi_0} \cup \{r_\alpha\}$. In all other cases we put $A^\alpha_\xi = \hat{A}^\alpha_\xi$ and our construction is completed.

Let $B_\xi = \bigcup_{\alpha<\mathfrak{c}} A^\alpha_\xi$ for $\xi < \kappa$. Then B_ξ is a Bernstein set thanks to the condition (2) and is not a 2-covering thanks to the conditions (5) and (6). The conditions (1) and (4) ensure us that the family $\{B_\xi : \xi < \kappa\}$ is a partition of $\mathbb{R}$ and the condition (3) makes this family a κ-S-covering. □

Remark 3. Let us observe that if κ is countable then the condition $2^\kappa \leq \mathfrak{c}$ is fulfilled. In general we need extra set theoretic assumptions. For example it is enough to assume Martin's Axiom, which implies that $2^\kappa = \mathfrak{c}$ for $\omega \leq \kappa < \mathfrak{c}$ (see [5]).

In more general situation, constructing S-coverings consisting of completely $\mathcal{I}$-nonmeasurable subsets of a given Polish group, none of which is a 2-covering is a bit more complicated. That is why we need some additional assumptions about a σ-ideal $\mathcal{I}$.

Theorem 3.3. *Let $(X, +)$ be an uncountable Abelian Polish group with a complete metric d. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal such that*

$$(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\forall \mathcal{D} \in [\mathcal{I}]^{<\mathfrak{c}})\, |B \setminus \bigcup \mathcal{D}| = \mathfrak{c}$$

and there exists $a \in \mathrm{range}(d)$, $a \neq 0$ such that

$$(\forall x \in X)\, \{y \in X : d(x, y) = a\} \in \mathcal{I}.$$

If κ is a cardinal number such that $2^\kappa = \mathfrak{c}$, then there exists a family $\{B_\xi : \xi < \kappa\}$ of pairwise disjoint subsets of X such that

(1) *$(\forall \xi < \kappa)\, B_\xi$ is a completely $\mathcal{I}$-nonmeasurable set,*
(2) *$(\forall \xi < \kappa)\, B_\xi$ is not a 2-covering,*
(3) *$\{B_\xi : \xi < \kappa\}$ is a κ-S-covering.*

Proof. Without loss of generality we can assume that $a = 1$. Let $\mathrm{Borel}(X) \setminus \mathcal{I} = \{P_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all $\mathcal{I}$-positive Borel subsets of X. Let us also enumerate the set $[X]^\kappa = \{F_\alpha : \alpha < \mathfrak{c}\}$. We proceed similarly as in the proof of Theorem 3.2, constructing a sequence $(\{A^\alpha_\xi : \xi < \kappa\})_{\alpha<\mathfrak{c}}$ of families of subsets of X of size less than continuum such that for every $\alpha < \mathfrak{c}$ the following conditions are fulfilled:

(1) for every different $\xi_1, \xi_2 < \kappa$ the sets $A^\alpha_{\xi_1}$ and $A^\alpha_{\xi_2}$ are disjoint;
(2) for every $\xi < \kappa$ the intersection $A^\alpha_\xi \cap P_\alpha$ is nonempty and we have $|A^\alpha_\xi| \leq |\alpha| \cdot \omega$;
(3) there exists $t_\alpha \in X$ such that $t_\alpha + F_\alpha \subseteq \bigcup_{\xi<\kappa} A^\alpha_\xi$ and for every $\xi < \kappa$ we have $|(t_\alpha + F_\alpha) \cap A^\alpha_\xi| = 1$;
(4) for every $\xi < \kappa$ and every $\beta < \alpha$ we have $A^\beta_\xi \subseteq A^\alpha_\xi$;
(5) for every $\xi < \kappa$ and every $x, y \in A^\alpha_\xi$ we have $d(x, y) \neq 1$.

Assume that we are at the αth step of the construction. Let $A_\xi = \bigcup_{\beta<\alpha} A^\beta_\xi$ and $A = \bigcup_{\xi<\kappa} A_\xi$. Moreover, let $C = \bigcup_{x\in F_\alpha} \bigcup_{a\in A}\{t \in X : d(t + x, a) = 1\}$. Then the set $T = C \cup (A - F_\alpha)$ is the set of "bad translations" of the set F_α. But C is a collection of less then continuum many unit spheres and $|A - F_\alpha| < \mathfrak{c}$ so according to our assumptions the complement of T is of size continuum. Thus we can choose $t_\alpha \notin T$.

Analogously, we can choose a subset $Y \subseteq P_\alpha$ of size κ such that

$$Y \cap ((t_\alpha + F_\alpha) \cup A \cup \{x \in X : (\exists a \in (t_\alpha + F_\alpha) \cup A)\, d(x, a) = 1\}) = \emptyset.$$

Finally, we enumerate sets $t_\alpha + F_\alpha = \{a_\xi : \xi < \kappa\}$ and $Y = \{b_\xi : \xi < \kappa\}$, put $A^\alpha_\xi = A_\xi \cup \{a_\xi, b_\xi\}$ for $\xi < \kappa$ and we are done.

Let $B_\xi = \bigcup_{\alpha<\mathfrak{c}} A^\alpha_\xi$ for $\xi < \kappa$. Then $\{B_\xi : \xi < \kappa\}$ is the needed family. □

Remark 4. Let us observe that in Theorem 3.3 we can replace the assumption

$$(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\forall \mathcal{D} \in [\mathcal{I}]^{<\mathfrak{c}})\, |B \setminus \bigcup \mathcal{D}| = \mathfrak{c}$$

by a stronger, but shorter assumption, namely $\mathrm{add}(\mathcal{I}) = \mathfrak{c}$.

When our Polish space is simply a Euclidean vector space and we deal with meagre or null sets, we can omit one assumption in Theorem 3.3.

Corollary 3.4. *Let $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$. Then for every cardinal number κ such that $2^\kappa = \mathfrak{c}$ there exists a family $\{B_\xi : \xi < \kappa\}$ of pairwise disjoint subsets of X such that*

(1) $(\forall \xi < \kappa)\, B_\xi$ *is a completely* $\mathcal{I}$*-nonmeasurable set,*
(2) $(\forall \xi < \kappa)\, B_\xi$ *is not a 2-covering,*
(3) $\{B_\xi : \xi < \kappa\}$ *is a* κ*-S-covering.*

Proof. It is enough to observe that we can repeat the proof of Theorem 3.3. Indeed, our choice of Y (and t_α) is possible because thanks to Lemma 1.6 after removing less than continuum many unit spheres from an $\mathcal{I}$-positive Borel set we have still continuum many points left. □

Corollary 3.4 remains true for every σ-ideal fulfilling conditions mentioned in Remark 2.

Just as in case of Theorem 3.2, assuming Martin's Axiom we obtain from Theorem 3.3 a suitable κ-S-covering for every $\kappa < \mathfrak{c}$. For example, we get a result concerning an S-covering made of Lebesgue completely nonmeasurable sets in $\mathbb{R}^n$.

Corollary 3.5. *Assume Martin's Axiom and* $\mathfrak{c} = \aleph_2$*. Then there exists a family* $\{B_\xi : \xi < \mathfrak{c}\}$ *of pairwise disjoint subsets of* $\mathbb{R}^n$ *such that*

(1) $(\forall \xi < \mathfrak{c})\, \lambda_*(B_\xi) = 0$ *and* $\lambda_*(\mathbb{R}^n \setminus B_\xi) = 0$*,*
(2) $(\forall \xi < \mathfrak{c})\, B_\xi$ *is not a 2-covering,*
(3) $\{B_\xi : \xi < \mathfrak{c}\}$ *is a* ω_1*-S-covering,*

where λ_* *denotes the inner Lebesgue measure in* $\mathbb{R}^n$*.*

Proof. Immediate from Theorem 3.3, Corollary 3.4 and Remark 4 together with the fact that under Martin's Axiom the additivity of the σ-ideal of Lebesgue null sets is equal to continuum. □

Theorem 3.3 gives us a κ-S-covering separately for every $\kappa < \mathfrak{c}$. It occurs that we can do this uniformly.

Definition 6. A family $\mathcal{A}$ of pairwise disjoint subsets of an uncountable Abelian Polish group $(X, +)$ is called a $<\kappa$*-S-covering*

$$(\forall F \in [X]^{<\kappa})(\exists t \in X)\Big(F + t \subseteq \bigcup \mathcal{A} \wedge (\forall A \in \mathcal{A})|(F + t) \cap A| \leq 1\Big).$$

Theorem 3.6. *Let* $(X, +)$ *be an uncountable Abelian Polish group with a complete metric* d*. Let* $\mathcal{I} \subseteq \mathcal{P}(X)$ *be a* σ*-ideal such that*

$$(\forall B \in \mathrm{Borel}(X) \setminus \mathcal{I})(\forall \mathcal{D} \in [\mathcal{I}]^{<\mathfrak{c}})\, |B \setminus \bigcup \mathcal{D}| = \mathfrak{c}$$

and there exists $a \in \mathrm{range}(d)$*,* $a \neq 0$ *such that*

$$(\forall x \in X)\, \{y \in X : d(x, y) = a\} \in \mathcal{I}.$$

If for every $\kappa < \mathfrak{c}$ *we have* $2^\kappa \leq \mathfrak{c}$ *then there exists a family* $\{B_\xi : \xi < \mathfrak{c}\}$ *of pairwise disjoint subsets of* X *such that*

(1) $(\forall \xi < \mathfrak{c})\, B_\xi$ *is a completely* $\mathcal{I}$*-nonmeasurable set,*
(2) $(\forall \xi < \mathfrak{c})\, B_\xi$ *is not a 2-covering,*
(3) $\{B_\xi : \xi < \mathfrak{c}\}$ *is a* $<\mathfrak{c}$*-S-covering.*

Proof. The construction is analogous to this from the proof of Theorem 3.3. □

4. I-coverings on the plane

In this chapter we focus our attention on the plane $\mathbb{R}^2$ treated as a Polish group. According to Definition 4 we can investigate a κ-covering as a subset of the plane such that every planary set of size κ can be translated into it. However, we may also generalize this definition letting sets of size κ to be not only translated but moved by any isometry.

Definition 7. We say that a set $A \subseteq \mathbb{R}^2$ is a κ-I-covering if

$$(\forall B \in [\mathbb{R}^2]^\kappa)(\exists \varphi : \mathbb{R}^2 \to \mathbb{R}^2)(\varphi \text{ is an isometry and } \varphi[B] \subseteq A).$$

It occurs that we cannot partition the plane into two sets none of which is a 2-I-covering.

Theorem 4.1. *If $\{A_0, A_1\}$ is a partition of $\mathbb{R}^2$ then one of the sets A_0, A_1 is a 2-I-covering.*

Proof. Suppose that A_0 is not a 2-I-covering. Then there exists a positive real d such that none two points in A_0 are at a distance of d from each other. Let us fix any $a \in A_0$ and consider a circle C with a center a and a radius equal to d. Next, let us fix a halfline that starts from a and consider such a sequence $(a_n)_{n<\omega}$ of elements of this halfline that $d(a, a_n) = (n + 2)d$ for all $n < \omega$. Then for every real $x \in [(n + 1)d, (n + 3)d]$ there exists a point $p \in C$ such that $d(p, a_n) = x$.

Observe now that $C \subseteq A_1$. Moreover, at least one of every two consecutive elements of the sequence $(a_n)_{n<\omega}$ belongs to A_1. Hence for every $x > 0$ we can find two elements of A_1 which are at a distance of x from each other. Consequently, the set A_1 is a 2-I-covering. □

Next two theorems show that from the point of view of Bernstein sets there is a big difference between 2-I-coverings and 3-I-coverings.

Theorem 4.2. *Every Bernstein set is a 2-I-covering.*

Proof. Let $B \subseteq \mathbb{R}^2$ be a Bernstein set. To show that B is also a 2-I-covering let us fix two different points $a, b \in \mathbb{R}^2$. It is enough to observe that any circle with a center in a fixed point $c \in B$ and a radius $d(a, b)$ (where d stands for a standard Euclidean metric) is a perfect set, thus meets B. □

Theorem 4.3. *There exists a Bernstein set which is not a 3-I-covering.*

Proof. Let $\mathrm{Perf}(\mathbb{R}^2) = \{P_\alpha : \alpha < \mathfrak{c}\}$ be a fixed enumeration of all perfect subsets of $\mathbb{R}^2$. We build by transfinite induction two sequences $(a_\alpha)_{\alpha<\mathfrak{c}}, (b_\alpha)_{\alpha<\mathfrak{c}}$ of elements of the plane satisfying the following conditions:

(1) $(\forall \alpha < \mathfrak{c})\, a_\alpha, b_\alpha \in P_\alpha$,
(2) $\{a_\alpha : \alpha < \mathfrak{c}\} \cap \{b_\alpha : \alpha < \mathfrak{c}\} = \emptyset$,
(3) $(\forall \alpha, \beta, \gamma < \mathfrak{c})(d(a_\alpha, a_\beta) \neq 1 \vee d(a_\alpha, a_\gamma) \neq 1 \vee d(a_\beta, a_\gamma) \neq 1)$.

Suppose that we have already constructed $(a_\xi)_{\xi<\alpha}$ and $(b_\xi)_{\xi<\alpha}$ for some $\alpha < \mathfrak{c}$. Since the set $A = \{(a_{\xi_1}, a_{\xi_2}) : \xi_1, \xi_2 < \alpha \wedge d(a_{\xi_1}, a_{\xi_2}) = 1\}$ has at most $|\alpha \times \alpha| < \mathfrak{c}$ elements and for every pair $(a_{\xi_1}, a_{\xi_2}) \in A$ there are only two points with distance 1 from both a_{ξ_1} and a_{ξ_2} we can pick $a_\alpha \in P_\alpha \setminus (\{a_\xi : \xi < \alpha\} \cup \{b_\xi : \xi < \alpha\})$ such that $d(a_\alpha, a_{\xi_1}) \neq 1$ or $d(a_\alpha, a_{\xi_2}) \neq 1$ for all $\xi_1, \xi_2 < \alpha$. Let b_α be any element of $P_\alpha \setminus (\{a_\xi : \xi \leq \alpha\} \cup \{b_\xi : \xi < \alpha\})$.

Let us put $B = \{a_\alpha : \alpha < \mathfrak{c}\}$. The condition (2) ensures B is a Bernstein set. To show that B is not a 3-I-covering it is enough to observe that there is no equilateral triangle of sides of length 1 with all vertices in B. □

When we replace Bernstein sets by completely $\mathcal{I}$-nonmeasurable sets then it occurs that the theorem analogous to Theorem 4.2 may not be true.

Theorem 4.4. *Let $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$. Then there exists a completely $\mathcal{I}$-nonmeasurable planary set which is not a 2-I-covering.*

Proof. Let $\mathrm{Borel}(X) \setminus \mathcal{I} = \{B_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all $\mathcal{I}$-positive Borel subsets of X. We build by transfinite induction two sequences $(a_\alpha)_{\alpha<\mathfrak{c}}, (b_\alpha)_{\alpha<\mathfrak{c}}$ of elements of the plane satisfying the following conditions:

(1) $(\forall \alpha < \mathfrak{c})\, a_\alpha, b_\alpha \in B_\alpha$,
(2) $\{a_\alpha : \alpha < \mathfrak{c}\} \cap \{b_\alpha : \alpha < \mathfrak{c}\} = \emptyset$,
(3) $(\forall \alpha, \beta < \mathfrak{c})\, d(a_\alpha, a_\beta) \neq 1$.

Assume that we are at an αth step of the construction. Let

$$D = B_\alpha \setminus \bigcup_{\beta<\alpha} \{a \in \mathbb{R}^2 : d(a, a_\beta) = 1\}.$$

From Lemma 1.6 we get $|D| = \mathfrak{c}$. Let us pick $a_\alpha \in D \setminus \{a_\beta : \beta < \alpha\}$ and let $b_\alpha \in B_\alpha \setminus (\{a_\beta : \beta \leq \alpha\} \cup \{b_\beta : \beta < \alpha\})$.

Finally, the set $B = \{a_\alpha : \alpha < \mathfrak{c}\}$ is completely $\mathcal{I}$-nonmeasurable and not a 2-I-covering. □

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