Wrocław, March 26, 2021

Exercise list no 5

Simulations and algorithmic applications of Markov chains

(Poincare constant). Let X be a reversible Markov chain with a transition matrix $\mathbf{P}$ on a graph $G = (V, K)$, where $V = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ are the vertices and $K = \{(\mathbf{e}_i, \mathbf{e}_j) : \mathbf{P}(\mathbf{e}_i, \mathbf{e}_j) > 0\}$ are the edges. For a directed edge $\tilde{k} = (\mathbf{e}_i, \mathbf{e}_j)$ ($\mathbf{e}_i$ is the starting edge) define $\Lambda(\tilde{k}) = \pi(\mathbf{e}_i)\mathbf{P}(\mathbf{e}_i, \mathbf{e}_j)$. For set verices $\mathbf{e}_i$ and $\mathbf{e}_j$ let $\Gamma(\mathbf{e}_i, \mathbf{e}_j)$ be a deterministic, unique path. Let $|\Gamma(\mathbf{e}_i, \mathbf{e}_j)| = \sum_{\tilde{k} \in \Gamma(\mathbf{e}_i, \mathbf{e}_j)} 1$ be a length of that path. Poincare constant is defined as

$$\gamma_P := \max_{\tilde{k}} \left\{ \frac{1}{\Lambda(\tilde{k})} \sum_{(\mathbf{e}_i, \mathbf{e}_j) : \tilde{k} \in \Gamma(\mathbf{e}_i, \mathbf{e}_j)} |\Gamma(\mathbf{e}_i, \mathbf{e}_j)| \pi(\mathbf{e}_i) \pi(\mathbf{e}_j) \right\}$$

(for set $\tilde{k}$ the sum goes through all the vertices $(\mathbf{e}_i, \mathbf{e}_j)$ such that the edge $\tilde{k}$ belongs to the path $\Gamma(\mathbf{e}_i, \mathbf{e}_j)$).

Theorem 1

Let λ_2 denote the second largest (in terms of absolute value) eigenvalue of a transition matrix $\mathbf{P}$ of a reversible Markov chain X. Then

$$|\lambda_2| \leq 1 - \frac{1}{\gamma_P}.$$

Similarly to the Poincare constant, for a reversible and ergodic Markov chain $X \sim \mathbf{P}$ one can define the Cheeger constant via

$$\gamma_C := \min_{A \subset E : \pi(A) \leq 1/2} \frac{\sum_{\mathbf{e}_i \in A} \sum_{\mathbf{e}_j \in A^C} \pi(\mathbf{e}_i) \mathbf{P}(\mathbf{e}_i, \mathbf{e}_j)}{\pi(A)}.$$

Theorem 2

With λ_2 defined as previously we have

$$|\lambda_2| \leq 1 - \frac{1}{2} \gamma_C^2.$$

Theorem 3

For a reversible Markov chain $X \sim \mathbf{P}$ we have

$$d_{TV}(\delta_{\mathbf{e}} \mathbf{P}^n, \pi) \leq \frac{1}{2} \frac{1}{\sqrt{\pi(\mathbf{e})}} |\lambda_2|^n,$$

where $\delta_{\mathbf{e}} \mathbf{P}^n$ is the distribution of the $n-$th step of the chain that started in state $\mathbf{e}$.

Theorem 4

For any (not necessarily reversible!) Markov chain $X \sim \mathbf{P}$ we have

$$d_{TV}(\delta_{\mathbf{e}}\mathbf{P}^n, \pi) \le \frac{1}{2}\frac{1}{\sqrt{\pi(\mathbf{e})}}|\lambda_2(M)|^{\frac{n}{2}},$$

where $\lambda_2(M)$ is the second largest (in terms of absolute value) eigenvalue of a matrix $M = \mathbf{P}\tilde{\mathbf{P}}$.

Exercise 1 For $\phi : E \to \mathcal{R}$ define $Var(\phi) = \sum_{\mathbf{e}} \phi^2(\mathbf{e})\pi(\mathbf{e}) - (\sum_{\mathbf{e}} \phi(\mathbf{e})\pi(\mathbf{e}))^2$. Show that $Var(\phi) = \frac{1}{2}\sum_{\mathbf{e}_i,\mathbf{e}_j \in E}(\phi(\mathbf{e}_i) - \phi(\mathbf{e}_j))^2\pi(\mathbf{e}_i)\pi(\mathbf{e}_j)$.

Exercise 2 Let $n(\varepsilon)$ be such that $d_{TV}(\delta_{\mathbf{e}}\mathbf{P}^{n(\varepsilon)}, \pi) \le \varepsilon$. Show that for $n_{\gamma_P}(\varepsilon) = \gamma_P \log\left(\frac{1}{2\varepsilon\sqrt{\pi(\mathbf{e})}}\right)$

$$d_{TV}(\delta_{\mathbf{e}}\mathbf{P}^{n_{\gamma_P}(\varepsilon)}, \pi) \le \varepsilon,$$

where γ_P is the Poincare constant.

Exercise 3 Show that for $n_{\gamma_C}(\varepsilon) = \frac{2}{\gamma_C^2}\log\left(\frac{1}{2\varepsilon\sqrt{\pi(\mathbf{e})}}\right)$

$$d_{TV}(\delta_{\mathbf{e}}\mathbf{P}^{n_{\gamma_C}(\varepsilon)}, \pi) \le \varepsilon,$$

where γ_C is the Cheeger constant.

Exercise 4 Consider the symmetric random walk on a circle: $E = \{0, 1, \dots, n-1\}$.

$$\mathbf{P} = \begin{bmatrix} 1-2p & p & 0 & 0 & \dots & 0 & 0 & p \\ p & 1-2p & p & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 1-2p & p & 0 & \dots & 0 & 0 \\ & & & & & \ddots & & \\ p & 0 & 0 & 0 & 0 & \dots & p & 1-2p \end{bmatrix},$$

where $p < \frac{1}{2}$. Calculate (or estimate) γ_P.

Exercise 5 For previous exercise, calculate (or estimate) γ_C.

Exercise 6 For what p the Cheeger constant is better that the Poincare constant in the case of random walk on a circle (by better we mean that it gives better estimation in Theorem 2).

Exercise 7 Once again consider a random walk on a circle: $E = \{0, 1, \dots, n-1\}$

$$\mathbf{P} = \begin{bmatrix} 1-p & p & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1-p & p & 0 & \dots & 0 & 0 \\ & & & & & \ddots & & \\ p & 0 & 0 & 0 & 0 & \dots & 0 & 1-p \end{bmatrix}.$$

Show that by calculating $\mathbf{M} = \mathbf{P}\tilde{\mathbf{P}}$ the task to calculate γ_C and γ_P reduces to exercises 4 and 5. Give the estimations.

Exercise 8 Let X be a simple random walk on a graph $G = (V, K)$, where $V = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ are the vertices and $K = \{(\mathbf{e}_i, \mathbf{e}_j) : \mathbf{P}(\mathbf{e}_i, \mathbf{e}_j) > 0\}$ are the edges. Let the transition matrix be $\mathbf{P}(v_i, v_j) = 1/d(v_i)$ if $(v_i, v_j) \in K$, where $d(v_i)$ is the degree of vertex v_i. Let $\Gamma(v, v')$ denote some choice of a path from v to v' which doesn't include the same edge more than once. Define

$$d^* = \max_v d(v), \qquad s^* = \max_{v,v'} |\Gamma(v, v')|, \qquad \eta^* = \max_{\tilde{k} \in K} \#\{(v, v') \in V^2 : \tilde{k} \in \Gamma(v, v')\}.$$

Show that $\gamma_P \leq \frac{(d^*)^2 s^* \eta^*}{|K|}$.