Wrocław, May 7, 2021

Exercise list no 9

Simulations and algorithmic applications of Markov chains

Let's recall briefly the hard-core model introduced in the lecture notes. We have graph $G = (V, K)$, where $V = \{v_1, \dots, v_M\}$ and K is the set of edges. To each vertix we assign spin -1 or $+1$ and each of those assignments we call a configuration. $\xi \in \{-1, +1\}^{|V|}$ is a valid configuration if no two vertices that are adjecent in graph G have both the value $+1$. In the lecture notes you can find the construction of MCMC (Gibbs sampler), which state space was the space of all valid configurations and the stationary distribution was the uniform distribution:

$$\pi_G(\xi) = \begin{cases} \frac{1}{Z_G} & \text{if } \xi \in PK \\ 0 & \text{othwerwise} \end{cases},$$

where Z_G is the normalizing constant, i.e. $Z_G = \sum_{\xi \in \{0,1\}^V} \mathbf{1}(\xi \in PK)$.

Exercise 1 Show that the Gibbs algorithm for the hard-core model is an irreducible Markov chain.

Exercise 2 Show that for every $v \in V$ conditional probability that v has spin $+1$, conditioned on the fact that all neighbours of v have spin -1, is equal to $\frac{1}{2}$.

Exercise 3 Consider the following generalized hard-core model. This model allows for varying intesities of spin $+1$ in the graph. We introduce a parameter $\lambda > 0$ and in this model every valid configuration has probability

$$\pi_{G,\lambda}(\xi) = \begin{cases} \frac{\lambda^{\eta(\xi)}}{Z_{G,\lambda}} & \text{if } \xi \in PK \\ 0 & \text{otherwise} \end{cases},$$

where $\eta(\xi)$ is the number of $+1$ spins in configuration ξ and $Z_{G,\lambda}$ is the normalizing constant, i.e. $Z_{G,\lambda} = \sum_{\xi \in \{0,1\}^V} \lambda^{\eta(\xi)} \mathbf{1}(\xi \in PK)$.
Show that for every $v \in V$ conditional probability that v has spin $+1$, conditioned on the fact that all neighbours of v have spin -1, is equal to $\frac{\lambda}{\lambda+1}$.

Exercise 4 Construct a MCMC algorithm for generalized hard-core model introduced in Exercise 3.

Exercise 5 Let $a = (a_1, \dots, a_n), a_i \in \mathbb{N}$ and $b \in \mathbb{N}$ be given. Consider the space of configurations $\{0, 1\}^n$. Configuration $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ is called valid if $a \cdot x := \sum_{i=1}^n a_i x_i \leq b$. Construct Markov chain for which the stationary distribution is the uniform distribution on all valid configurations, i.e.

$$\pi(x) = \begin{cases} \frac{1}{Z} & \text{if } x \text{ is valid configuration} \\ 0 & \text{otherwise} \end{cases}.$$

Remark: (Knapsack problem) If vector a denotes the weights of n items and b is the maximum capasity of the knapsack, then Z can be interpreted as the number of possible combinations of items that can be put into the knapsack.

Exercise 6 Give Metropolis algorithm for knapsack problem.

Exercise 7 Let $G = (V, K)$ be a connected graph with $|V| = N$ and let X be a uniform random variable on a set $\{1, 2, \ldots, q\}^N$, i.e. we choose random coloring from all the possible colorings of the graph. Prove that the probability that X is valid $q-$coloring of the graph can be bounded from above by

$$\left(\frac{q-1}{q}\right)^{N-1}.$$

Exercise 8 Let $G = (V, K)$ be any connected graph, where maximal degree of any vertex is equal to d^*. Let $q = d^* + 1$. Show that then exists at least one valid $q-$coloring of the graph. Show that in this case the Gibbs algorithm for $q-$coloring is not necessary irreducible.

Exercise 9 Let's change q from previous exercise to be $q = d^* + 2$. Show that in this case the Gibbs algorithm for $q-$coloring is irreducible.