Wrocław, April 23, 2021

Exercise list no 7

Simulations and algorithmic applications of Markov chains

If something is called random without specifing the underlying distribution, you should assume that the distribution is uniform.

Exercise 1 *C*onsider graph $G = (V, E)$ with $V = \{1, 2, \ldots, N\}$ and $E = \{(i, i+1) : i \in \{1, 2, \ldots, N-1\}\}$. Show that in a hard-core model on this graph (one-dimensional hard-core model) the number of all valid configurations is equal to f_{N+1}, where f_k are Fibonacci numbers.

Exercise 2 *H*ow many valid q-colorings exist for the graph from Exercise 1?

Exercise 3 *H*ow many valid q-colorings exist for a full binary tree of depth n?

Exercise 4 *P*rove the following Lemma.

Lemma 1
Let $\varepsilon \in [0, 1]$, k be a positive natural number and $a_1, \ldots, a_k$, $b_i, \ldots, b_k$ be non-negative numbers such that

$$\left(1 - \frac{\varepsilon}{2k}\right) \le \frac{a_j}{b_j} \le \left(1 + \frac{\varepsilon}{2k}\right), \qquad j = 1, \ldots, k.$$

Then

$$1 - \varepsilon \le \frac{a}{b} \le 1 + \varepsilon,$$

where $a = \prod_{j=1}^k a_j$, $b = \prod_{j=1}^k b_j$.

Exercise 5 *L*et $R = (R^1, \ldots, R^q)$ be a random permutation of the set $\{1, 2, \ldots, q\}$. Set $S \subset \{1, \ldots, q\}$ and define

$$X = R^i, \qquad \text{where} \qquad i = \min_j \{j : R^j \in S^C\},$$

S^C is complement of S. Show that $X \sim$Unif(S^C), i.e. X is distributed uniformly on S^C.

Exercise 6 *L*et X and X' be two q-colorings of the graph $G = (V, K)$. Set a vertex $v \in V$ and let $S(X(v))$ be a set of colors of neighbours of v in coloring X. Define

$$B_2 = \{r \in \{1, \ldots, q\} : r \in S(X(v)) \cap S(X'(v))\},$$

$$B_1 = \left\{r \in \{1, \ldots, q\} : r \in \left[S(X(v)) \cap (S(X'(v)))^C\right] \cup \left[((S(X(v))^C \cap S(X'(v)))\right]\right\},$$

$$B_0 = \left\{r \in \{1, \ldots, q\} : r \in \left[S(X(v)) \cup S(X'(v))\right]^C\right\}.$$

Let $R = (R^1, \ldots, R^q)$ be a random permuation of the set $\{1, 2, \ldots, q\}$. Further define

$$Y = R^a, \qquad \text{where} \qquad a = \min_j \{j : R^j \in S(X(v))^C\}$$

and

$$Z = R^b, \qquad \text{where} \qquad b = \min_j \{j : R^j \in S(X'(v))^C\}.$$

Show that $P(Y = Z) = \frac{|B_0|}{|B_0|+|B_1|}$, where $|B_i|$ is the power of set B_i.

Exercise 7 *D*escribe a simple and effective method of generating a random permutation of the set $\{1, 2, \ldots, d\}$.

Exercise 8 *C*onsider Gibbs algorithm for q-colorings of the graph (the version with randomization of verices, not the 'sweep' one). Let $G = (V, E)$, $V = \{v_1, \ldots, v_M\}$. Show that for all $v \in V$ probability that v will be chosen during the first M iterations of the algorithm is equal to at least $1 - \frac{1}{e}$.

Exercise 9 *F*or $G = (V, E)$ subset $I \subset V$ is called a independent set in G if none of the vertices in I are connected by an edge from G. Let $\mathcal{I}(G)$ denote the set of all independent sets in G. Give an example of a Markov chain, which has uniform stationary distribution on $\mathcal{I}(G)$.
Hint: Construct Gibbs algorithm, which picks a random vertex and considers removing or adding it to current independent set/state.