Wrocław, April 16, 2021

# Exercise list no 6

## Simulations and algorithmic applications of Markov chains

Again, lets recall few definitions from the lecture

**Def 1**

Let $X$ be an ergodic Markov chain with a state space $E$, a transition matrix $\mathbf{P}$, an initial distribution $\nu$ and a stationary distribution $\pi$. We call random variable $T$ (that depends on initial distribution $\nu$!) a Strong Stationary Time (SST) if it is a stopping time and

$$\forall \mathbf{e} \in E : \quad P(X_k = \mathbf{e} | T = k) = \pi(\mathbf{e}).$$

**Def 2**

For two measures $\mu, \nu$ on the state space $E$ we define the seperation distance to be

$$sep(\mu, \nu) = \max_{\mathbf{e} \in E} \left(1 - \frac{\mu(\mathbf{e})}{\nu(\mathbf{e})}\right).$$

**Lemma 1**

$d_{TV}(\mu, \nu) \leq sep(\mu, \nu)$

**Theorem 1**

If $T$ is SST for Markov chain $X$, then

$$sep(\mu \mathbf{P}^n, \pi) \leq P(T > n).$$

**Def 3**

For a complete graph $G$ with $2n$ vertices we define matching to be the partition of those vertices into $n$ disconnected pairs.

**Exercise 1** Prove Lemma 1.

**Exercise 2** How many matching are there in a graph with $2n$ vertices?

**Exercise 3** From the lecture recall how to construct SST for matchings. We have the following equality in distribution

$$T = \sum_{i=1}^{n} X_i,$$

where $X_i$ are independent random variables with distribution $Geo(p_i)$ with some unknown parameter $p_i$. Find that parameter.

**Exercise 4** Calculate $ET$ and $VarT$ from Exercise 3 and prove that

$$s(\mu\mathbf{P}^{n\log n+cn},\pi)\leq P(T>n\log n+cn)\leq\frac{1}{c^2}.$$

**Exercise 5** (Random walk on $d-$dimensional cube) Consider the state space $E=\{0,1\}^d$. In each step we choose uniformly one of the coordinates and toss a (symmetrical) coin. If the coin shows up to be head, set the chosen coordianate to 1, if it shows up to be tails, set the chosen coordinate to 0. Find SST and give for which $n$ separation distance (using Theorem 1) is small.

**Exercise 6** We have $n$ cards. In each step, we independently add 0 or 1 to each card with equal probability. After $k$ steps, each card has $k$ numbers written on it. Let $T$ be the first moment when all $n$ series of numbers are different. Show that

$$P(T\leq t)\approx exp\left(-\frac{n^2}{2^t}\right)\qquad\qquad(*).$$

Use that $1-\frac{i}{2^t}\approx exp\left(-\frac{i}{2^t}\right)$.

**Remark 1**

Above random variable $T$ is SST for riffle shuffle. $(*)$ gives us that

$$P(T>t)\approx 1-exp\left(-\frac{n^2}{2^t}\right)$$

and it is small when $t\gg 2\log n$ or large (close to 1) when $t\ll 2\log n$.

**Exercise 7** Prove Lemma 9 from the lecture notes (equivalence of the three shuffling methods).