Wrocław, March 12, 2021

Exercise list no 4

Simulations and algorithmic applications of Markov chains

Exercise 1 (Simple random walk) Let X be a Markov chain on a state space $E = \{0, 1, \ldots, N\}$ with the following transition matrix

$$\mathbf{P} = \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ q_1 & r_1 & p_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & q_i & r_i & p_i & 0 \ldots & 0 \\ & & & & \ddots & \ddots & \ddots & \\ 0 & \ldots & & & & q_{N-1} & r_{N-1} & p_{N-1} \\ 0 & \ldots & & & & 0 & q_N & r_N \end{bmatrix},$$

where $q_0 = p_N = 0$ and for all $i \in \{1, \ldots, N-1\}$: $p_i > 0, q_i > 0, p_i + q_i + r_i = 1$. Find the stationary distribution of this chain (on list 2, exercise 8 we had a special case of this chain with $p_0 = q_N = 1$).

Exercise 2 (Simple random walk on a circle) Let X be a Markov chain on a state space $E = \{0, 1, \ldots, N\}$ with the following transition matrix

$$\mathbf{P} = \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & 0 & \ldots & q_0 \\ q_1 & r_1 & p_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & 0 & \ldots & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & q_i & r_i & p_i & 0 \ldots & 0 \\ & & & & \ddots & \ddots & \ddots & \\ 0 & \ldots & & & & q_{N-1} & r_{N-1} & p_{N-1} \\ p_N & \ldots & & & & 0 & q_N & r_N \end{bmatrix},$$

where for all $i \in \{0, \ldots, N\}$: $p_i > 0, q_i > 0, p_i + q_i + r_i = 1$. In other words, this is a simple random walk on a circle - when we are in state i we can go to $i - 1.i, i + 1$ and we have that $N + 1 \equiv 0$ oraz $0 - 1 \equiv N$.

- What conditions have to be fullfiled for parameters $p_i, q_i, i = 0, \ldots, N$, so that chain X is reversible?

- In cases where X is reversible find the stationary distribution.

Exercise 3 (Generating random matrix with set sum in each row and column) Below you can see an example of 4×4 matrix with set sum in each row and each column.

	110	**274**	**84**	**118**
210	67	112	26	5
93	19	52	18	4
217	21	88	23	85
66	3	22	17	24

Let E be a set of 4×4 matrices with natural elements in which each row and each column sums up to the numbers given in the example above. Find a method to generate a uniform distribution on E (find a Markov chain with a stationary distribution being uniform on E).

Exercise 4 Consider n books labeled from 1 to n. Books are placed on a shelf in one row, in position from 1 (first one on the left) to n. In each move one chooses a book with some number b with probability p_b, $0 < p_i < 1, i = 1, \ldots, n, \sum_{i=1}^{n} p_i = 1$. If the chosen book is on position 1, nothing happens. Otherwise, if you have chosen a book on position $k > 1$, then swap this book with a book on position $k - 1$. Described process is a Markov chain, let's call it X. Consider a state space where a state is described via $\mathbf{e} = (e_1, \ldots, e_n)$, where e_b is position of book with number b. Find the transition matrix of Markov chain X and show that

$$\pi(\mathbf{e}) = \frac{1}{C} \prod_{r=1}^{n} p_r^{-e_r},$$

is a stationary distribution of this chain, where $C > 0$ is a normalizing constant.

Exercise 5 (Coupon collector problem) At one step we draw a single coupon out of n coupons, independently from other drawings and uniformly at random (i.e., each coupon has probability $1/n$ of appearing). Let T be the first moment when each coupon was drawn at least once. Show that

$$P(T > k) \leq e^{-c} \quad \text{for} \quad k = n \log n + cn.$$