Wrocław, June 8, 2021

Exercise list no 11

Simulations and algorithmic applications of Markov chains

Definition 1
We call Z_G an $(\epsilon, \delta)-$approximation scheme for Z if

$$P(|Z - Z_G| \geq \epsilon Z_G) \leq \delta.$$

Definition 2
Markov chain on a state space $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ with the transition matrix $\mathbf{P}_X$ is Möbius monotone w.r.t. partial ordering $\preceq$ if $\mathbf{C}^{-1}\mathbf{P}_X\mathbf{C} \geq 0$ (component-wise), where $\mathbf{C}(\mathbf{e}, \mathbf{e}') = 1$ if $\mathbf{e} \preceq \mathbf{e}'$ and 0 otherwise.

Exercise 1 Prove the following lemma from the lecture:

Lemma 1
Let $Y_1, \ldots, Y_R$ be i.i.d with $P(Y_1 = 1) = 1 - P(Y_1 = 0) = p$. Let $\hat{Y}_R = \frac{1}{R}\sum_{i=1}^{R} Y_i$ be the estimator of the mean. If further $R > \frac{3\log(\frac{2}{\delta})}{\epsilon^2 p}$, then

$$P(|\hat{Y}_R - p| > \epsilon p) \leq \delta.$$

Exercise 2 Let $\widetilde{s}_i$ be an $(\frac{\epsilon}{2m}, \frac{\delta}{m})-$approximation scheme for s_i. Show that

$$\widetilde{z} = \widetilde{s}_1 \cdot \widetilde{s}_2 \cdot \ldots \cdot \widetilde{s}_m$$

is an $(\epsilon, \delta)-$approximation scheme for $z = s_1 \cdot s_2 \cdot \ldots \cdot s_m$.

Exercise 3 (Example 3.2 *"Cat Eats Mouse Eats Cheese"* from the book: P. Bremaud *Markov chains: Gibbs fields, Monte Carlo simulation, and queues*, 1999.)
The mouse moves through the rooms (Fig. 1) with uniform probability (i.e. if it is in a room with 2 neighbouring rooms, it will move to any of them with probability $\frac{1}{2}$). If is gets to the room 5 (where the cheese is), it will stay there, but if it gets to the room 3 (where the cat is) it will get eaten. Calculate the probability of getting to the room 5 before getting to the room 3 starting from the rooms $i = 1, 2, 3, 4, 5$.
Note: if it was on the last lecture (09.06.2020), you should use Siegmund duality, because it will be easier - but you can also do it without it.

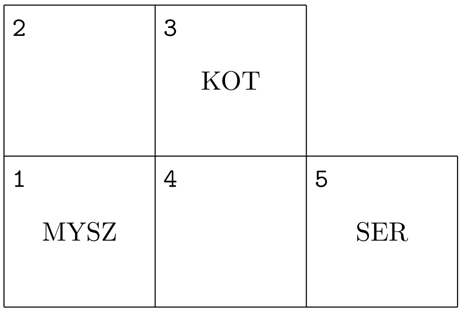

Figure 1: Maze, mouse, and murder

Exercise 4 Let $E = \{0,1\}^d$ and $\mathbf{e} = (e_1, \ldots, e_d) \in E$, i.e. $\mathbf{e}_i \in \{0,1\}$. Let's introduce a partial ordering $\mathbf{e} \preceq \mathbf{e}' \iff e_i \leq e'_i, i = 1, \ldots, d$. Let $\mathbf{C}$ be a matrix of the ordering $\preceq$ of dimension $|E| \times |E|$, i.e. $\mathbf{C}(\mathbf{e}, \mathbf{e}') = 1$ if $\mathbf{e} \preceq \mathbf{e}'$ and 0 otherwise. Show that

$$\mathbf{C}^{-1}(\mathbf{e}, \mathbf{e}') = \begin{cases} (-1)^{|\mathbf{e}'|-|\mathbf{e}|} & \text{if } \ \mathbf{e} \preceq \mathbf{e}', \\ 0 & \text{otherwise,} \end{cases},$$

where $|\mathbf{e}| = \sum_{i=1}^{d} e_i$.

Exercise 5 Show that the Möbius monotonicity definition is equivalent to

$$\forall(\mathbf{e}_i, \mathbf{e}_j \in E) \quad \sum_{\mathbf{e} \succeq \mathbf{e}_i} \mu(\mathbf{e}_i, \mathbf{e}) \mathbf{P}_X(\mathbf{e}, \{\mathbf{e}_j\}^{\downarrow}) \geq 0,$$

where $\{\mathbf{e}_j\}^{\downarrow} = \{\mathbf{e} : \mathbf{e} \preceq \mathbf{e}_j\}$ and $\mathbf{P}_X(\mathbf{e}, A) = \sum_{\mathbf{e}' \in A} \mathbf{P}_X(\mathbf{e}, \mathbf{e}')$.

Exercise 6 Show that if the partial ordering $\preceq$ is in fact a linear ordering, then the Möbius monotonicity is equivalent to stochastic monotonicity.