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Parisian ruin probability for two-dimensional Brownian risk model

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ABSTRACT

Let $(W_1(s), W_2(t)), s, t \geq 0$ be a bivariate Brownian motion with standard Brownian motion marginals and constant correlation $\rho \in (-1, 1)$. Parisian ruin is defined as a classical ruin that happens over an extended period of time, the so-called time-in-red. We derive exact asymptotics for the non-simultaneous Parisian ruin of the company conditioned on the event of non-simultaneous ruin happening. We are interested in finding asymptotics of such problem as $u \to \infty$ and with the length of time-in-red being of order $\frac{1}{u^2}$, where u represents initial capital for the companies. Approximation of this problem is of interest for the analysis of Parisian ruin probability in bivariate Brownian risk model, which is a standard way of defining prolonged ruin models in the financial markets.

1. Introduction

Consider the following Brownian risk model for two portfolios

$$R_i(t) = u_i + c_i t - W_i(t), i = 1, 2,$$

where the claims $W_i(t), t \geq 0$ are represented by two dependent standard Brownian motions, $u_i > 0$ can be interpreted as initial capitals and c_i can be interpreted as premiums. The following representation of the dependence between the claims has been proposed in Delsing et al. (2018) and Dębicki et al. (2020)

$$(W_1(s), W_2(t)) = (B_1(s), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)), \quad s, t \geq 0,$$

where B_1, B_2 are two independent standard Brownian motions and $\rho \in [-1, 1]$. The ruin probability of a single portfolio in the time horizon $[0, T], T > 0$ is given by (see e.g., Dębicki and Mandjes (2015))

$$\pi_T(c_i, u) := \mathbb{P}\left\{\inf_{t\in[0,T]} R_i(t) < 0\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]} W_i(t) - c_i t > u\right\}$$
$$= \Phi\left(-\frac{u}{\sqrt{T}} - c_i\sqrt{T}\right) + e^{-2c_i u}\Phi\left(-\frac{u}{\sqrt{T}} + c_i\sqrt{T}\right)$$

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for $i = 1, 2$, $u \geq 0$, with Φ the distribution function of an $N(0, 1)$ random variable. Since from self-similarity of Brownian motion we have the following equalities in distribution for $c_1' = \frac{c_1}{\sqrt{T}}$, $u' = \frac{u}{\sqrt{T}}$

$$B(tT) - c_1 t > u \Leftrightarrow \sqrt{T}B(t) - c_1 t > u \Leftrightarrow B(t) - c_1' t > u',$$

then without loss of generality one can assume $T = 1$. There are at least two different approaches on how to define the extension of the above to the two-dimensional model. Denote $W_i^*(s) = W_i(s) - c_i s$, $B_i^*(s) = B_i(s) - c_i s$, $i = 1, 2$. Define the simultaneous ruin probability as

$$\overline{\pi}_{A,\rho}(c_1, c_2, u, v) = \mathbb{P}\left\{\exists_{s\in A} : W_1^*(s) > u, W_2^*(s) > v\right\}$$

which has been recently studied in Dębicki et al. (2020) for $A = [0, 1]$. Similarly, define non-simultaneous ruin probability as

$$\pi_{A\times B,\rho}(c_1, c_2, u, v) = \mathbb{P}\left\{\exists_{s\in A, t\in B} : W_1^*(s) > u, W_2^*(t) > v\right\}$$

which has been studied for the case $A = B = [0, 1]$ in Dębicki et al. (2021). In this contribution we focus on an extension of the non-simultaneous results of ruin for two-dimensional risk portfolios. In Loeffen et al. (2013), Czarna and Renaud (2016) the so-called Parisian ruin of a single portfolio was investigated which is defined as

$$\mathcal{P}^*_{A,H(u)}(c, u) := \mathbb{P}\left\{\exists_{s'\in A}\forall_{s\in[s',s'+H(u)]}\chi(s) > u\right\},$$

for some $H(u) \geq 0$, $A = [0, T]$ and χ a Lévy process. This model defines the concept of the ruin as crossing the barrier over the extended period of time, the so-called time in red. It seems more natural than the classical ruin approach, since it allows for easier practical investigations whether the ruin has occurred. This model has also been studied for various sets A and various processes in many other contributions, e.g. Dębicki et al. (2015), Bai and Luo (2017), Dębicki et al. (2016), Dassios and Wu (2008). To analyse the model in two-dimensional framework we use the following definition of the ruin probability

$$\mathcal{P}^*_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, v) := \mathbb{P}\left\{\exists_{s'\in A, t'\in B}\forall_{s\in[s',s'+H_1(u)]}\forall_{t\in[t',t'+H_2(u)]}W_1^*(s) > u, W_2^*(t) > v\right\},$$

for some $H_1(u), H_2(u) \geq 0$ and intervals A, B. We refer to Dassios and Wu (2011), where one can find an application of Parisian ruin to actuarial risk theory, where R_i is treated as a surplus process of an insurance company with initial capital u_i. Similar model for simultaneous type of ruin has been investigated in Kriukov (2020).

For more general intervals A, B we have the following comparison between Parisian and classical ruin

$$\pi_{A\times B,\rho}(c_1, c_2, u, au) \geq \mathcal{P}^*_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au). \tag{1.1}$$

Further let

$$\mathscr{P}^*_{[0,1]^2,\boldsymbol{H}(u)}(c_1, c_2, u, au) := \mathbb{P}\left\{\exists_{s',t'\in[0,1]}\forall_{s\in[s',s'+H_1(u)]}\forall_{t\in[t',t'+H_2(u)]} \begin{array}{c} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \middle| \exists_{v,w\in[0,1]} \begin{array}{c} W_1^*(v) > u \\ W_2^*(w) > au \end{array}\right\}.$$

To simplify notation we denote

$$\mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) := \mathscr{P}^*_{[0,1]^2, \frac{(S_1,S_2)}{u^2}}(c_1, c_2, u, au)$$

and write

$$\mathcal{P}_{S_1,S_2}(c_1, c_2, u, au) := \mathcal{P}^*_{[0,1]^2, \frac{(S_1,S_2)}{u^2}}(c_1, c_2, u, au).$$

Similarly to the classical ruin, the Parisian ruin probability cannot be determined explicitly for general Gaussian risks, therefore our aim is to investigate the asymptotic behaviour as $u \to \infty$. Notice that

$$\mathscr{P}^*_{[0,1]^2, \frac{(S_1,S_2)}{u^2}}(c_1, c_2, u, au) = \frac{\mathcal{P}^*_{[0,1]^2, \frac{(S_1,S_2)}{u^2}}(c_1, c_2, u, au)}{\pi_{[0,1]^2,\rho}(c_1, c_2, u, au)}.$$

Together with results for $\pi_{[0,1]^2,\rho}(c_1, c_2, u, au)$ from Dębicki et al. (2021) calculating asymptotics of $\mathscr{P}_{S_1,S_2}(c_1, c_2, u, au)$ and $\mathcal{P}_{S_1,S_2}(c_1, c_2, u, au)$ is equivalent and to show a direct comparison between different kinds of ruin we decided to present results in terms of the former, but we focus the proofs on the latter. We investigate for which $\boldsymbol{H}(u)$ we have that for some $C > 0$

$$\lim_{u\to\infty} \mathscr{P}^*_{[0,1]^2,\boldsymbol{H}(u)}(c_1, c_2, u, au) = C.$$

We prove that the above is true for $\boldsymbol{H}(u) = (\frac{S_1}{u^2}, \frac{S_2}{u^2}) := \frac{(S_1,S_2)}{u^2}$ for some $S_1, S_2 > 0$. For the choice of $H_i(u) = o(\frac{1}{u^2})$, $\forall_{i\in\{1,2\}}$ following the same line of proof we have that

$$\lim_{u\to\infty} \mathscr{P}^*_{[0,1]^2,\boldsymbol{H}(u)}(c_1, c_2, u, au) = 1.$$

On the other hand, if we choose $\boldsymbol{H}(u)$ such that $\exists_{i\in\{1,2\}} u^2 H_i(u) \to \infty, H_i(u) < 1$, then the methods employed in this contribution are not sufficient and the asymptotics are of different order, even in the one-dimensional setting.

2. Main results

Based on the relation between a and ρ, either both of the coordinates impact the asymptotics, or one of the coordinates is negligible (up to a constant). We begin with cases where one of the coordinates dominates the other one and hence the results can be derived from one-dimensional models. Note that without loss of generality we can assume $a \le 1$. Additionally we can assume $a > 0$, since for $a < 0$ we already have $W_2(0) > au$ and hence the second coordinate vanishes. For the special case $a = \rho = 1$ notice that the only difference in coordinates comes from a drift and since

$$\mathbb{P}\{B(t) - c_1 t > u, B(t) - c_2 t > u\} = \mathbb{P}\{B(t) - \max(c_1, c_2)t > u\}$$

we can omit this case. Denote by Ψ the survival function of a standard Normal random variable.

Let $C_{\mathcal{P}}(S) = \mathbb{E}\left\{\exp\left(\sup_{t\ge 0}\inf_{s\in[0,\frac{S}{2}]}\sqrt{2}B(t-s) - 2|t-s|\mathbf{1}(t>s)\right)\right\}$, which by Dębicki et al. (2016)[Cor 3.5] is positive and finite.

Theorem 2.1. *If $a \le \rho$, then*

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = \frac{C_{\mathcal{P}}(S_1)}{2}.$$

Our next results are separated into different cases, based on a relative relation between ρ and $A_a = \frac{1}{4a}(1 - \sqrt{8a^2+1})$. Function A_a has been found by analytical calculations. Heuristically, when $\rho < 0$ is relatively big compared to a (in terms of absolute value), then it is less likely that the ruin will occur simultaneously and the asymptotics should be significantly different from the ones that have been discovered for simultaneous ruin in Kriukov (2020).

Denote $t_* = \frac{a}{\rho(2a\rho - 1)}$ and introduce the following notation for the constants

$$\mathcal{P}(w_1, w_2, S) := \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s'\in[0,\infty)}\forall_{s\in[s',s'+S[} : B(s) - w_1 s > x\right\} e^{w_2 x} dx,$$

$$\mathcal{H}(w_1, w_2, S) := \lim_{\Delta\to\infty}\int_{\mathbb{R}} \frac{1}{\Delta}\mathbb{P}\left\{\exists_{t'\in[0,\Delta]}\forall_{t\in[t',t'+S]} B(t) - w_1 t > x\right\} e^{w_2 x} dx,$$

$$\mathcal{R}(S_1, S_2) = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{s',t'\in[0,\infty)}\forall_{s\in[s',s'+S_1],t\in[t',t'+S_2]} : \begin{array}{c} W_1(s) - s > x \\ W_2(t) - at > y \end{array}\right\} e^{\frac{1-a\rho}{1-\rho^2}x + \frac{a-\rho}{1-\rho^2}y} dxdy \in (0, \infty).$$

Theorem 2.2. *Let $\rho \in (-1, 1)$ and $a \in (\max(0, \rho), 1]$ be given.*

(i) If $\rho > A_a$, then

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = \frac{\mathcal{R}(S_1, S_2)}{\mathcal{R}(0, 0)}. \tag{2.1}$$

(ii) If $\rho = A_a$ and $a < 1$, then

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = \frac{(1 - a\rho)\mathcal{P}(\frac{1-a\rho}{1-\rho^2}, \frac{1-a\rho}{1-\rho^2}, S_1)\mathcal{H}(a, 2a, S_2)}{2a(1-\rho^2)}. \tag{2.2}$$

(iii) If $\rho = A_a$, $a = 1$, then

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = \frac{C_{4,1}C'_{4,1} + C_{4,2}C'_{4,2}}{C_4}, \tag{2.3}$$

where $C_{4,1} = \mathcal{P}(2, 2, S_1)\mathcal{H}(1, 2, S_2)$, $C_{4,2} = \mathcal{P}(2, 2, S_2)\mathcal{H}(1, 2, S_1)$ and

$$C'_{4,1} = \begin{cases} e^{-2\frac{(\frac{1}{2}c_1 + c_2)^2}{3}}\Phi\left(c_2 + \frac{1}{2}c_1\right), & -\frac{1}{2}c_1 < c_2 \\ 1, & otherwise, \end{cases} \quad C'_{4,2} = \begin{cases} e^{-2\frac{(\frac{1}{2}c_2 + c_1)^2}{3}}\Phi\left(c_1 + \frac{1}{2}c_2\right), & -\frac{1}{2}c_2 < c_1 \\ 1, & otherwise, \end{cases}$$

$$C_4 = \begin{cases} e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\left(c_2 + \frac{1}{2}c_1\right) + e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\left(c_1 + \frac{1}{2}c_2\right), & c_2 > \max(-\frac{1}{2}c_1, -2c_1) \\ e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\left(c_2 + \frac{1}{2}c_1\right) + \frac{1}{2}, & -\frac{1}{2}c_1 < c_2 \le -2c_1 \\ \frac{1}{2} + e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\left(c_1 + \frac{1}{2}c_2\right), & -2c_1 < c_2 \le -\frac{1}{2}c_1 \\ 1, & c_2 \le \min(-\frac{1}{2}c_1, -2c_1). \end{cases}$$

(iv) If $a < 1$, $\rho < A_a$, then

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = -\frac{\mathcal{P}(\frac{1-a\rho}{1-\rho^2 t_*}, \frac{1-a\rho}{1-\rho^2 t_*}, S_1)\mathcal{H}(\frac{a}{t_*}, \frac{2a}{t_*}, S_2)}{2\rho}. \tag{2.4}$$

(v) If $a = 1$, $\rho < A_a$, then

$$\lim_{u\to\infty} \mathscr{P}_{S_1,S_2}(c_1, c_2, u, au) = -\frac{C_5}{2\rho}, \tag{2.5}$$

where $t_* = \frac{1}{\rho(2\rho-1)}$, $C_5 = \begin{cases} \mathcal{P}(\frac{1-\rho}{1-\rho^2 t_*}, \frac{1-\rho}{1-\rho^2 t_*}, S_1)\mathcal{H}(\frac{1}{t_*}, \frac{2}{t_*}, S_2) & c_1 \le c_2 \\ \mathcal{P}(\frac{1-\rho}{1-\rho^2 t_*}, \frac{1-\rho}{1-\rho^2 t_*}, S_2)\mathcal{H}(\frac{1}{t_*}, \frac{2}{t_*}, S_1), & c_1 > c_2. \end{cases}$

3. Proofs

3.1. Proof of Theorem 2.1

We divide the proof into two parts: $a < \rho$ and $a = \rho$, since the methods used are quite different. Further define $S_{1,2} = \max(S_1, S_2)$, which will be commonly used notation in both parts of the proof.

Case (i) : $a < \rho$. First note that

$$\mathcal{P}_{S_1,S_2}(c_1, c_2, u, au) \le \mathbb{P}\left\{\exists_{s'\in[0,1]}\forall_{s\in[s',s'+\frac{S_1}{u^2}]} W_1^*(s) > u\right\}.$$

On the other hand

$$\begin{aligned}
&\mathcal{P}_{S_1,S_2}(c_1, c_2, u, au)\\
&\ge \mathbb{P}\left\{\exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u, \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) - c_2 t > au \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}}, \rho B_1(t) + \sqrt{1-\rho^2}B_2(t) - c_2 t > au \end{array}\right\}\\
&\ge \mathbb{P}\left\{\exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u, \rho(u + c_1 t) + \sqrt{1-\rho^2}B_2(t) - c_2 t > au \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}}, \rho(u - \frac{1}{\sqrt{u}} + c_1 t) + \sqrt{1-\rho^2}B_2(t) - c_2 t > au \end{array}\right\}\\
&= \mathbb{P}\left\{\exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u, \sqrt{1-\rho^2}B_2(t) > (a-\rho)u + (c_2 - \rho c_1)t \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}}, \sqrt{1-\rho^2}B_2(t) > (a-\rho)u + \frac{\rho}{\sqrt{u}} + (c_2 - \rho c_1)t \end{array}\right\}\\
&\ge \mathbb{P}\left\{\forall_{s\in[0,1]} B_2(s) > \frac{(a-\rho)u + (c_2-\rho c_1)s + \frac{\rho}{\sqrt{u}}}{\sqrt{1-\rho^2}}, \exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1(t) - c_1 t > u \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}} \end{array}\right\}\\
&= \mathbb{P}\left\{\exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}} \end{array}\right\} \mathbb{P}\left\{\forall_{s\in[0,1]} B_2(s) > \frac{(a-\rho)u + (c_2-\rho c_1)s + \frac{\rho}{\sqrt{u}}}{\sqrt{1-\rho^2}}\right\}.
\end{aligned}$$

Since $a < \rho$, we have that

$$\lim_{u\to\infty} \mathbb{P}\left\{\forall_{s\in[0,1]} B_2(s) > \frac{(a-\rho)u + (c_2-\rho c_1)s + \frac{\rho}{\sqrt{u}}}{\sqrt{1-\rho^2}}\right\} = 1.$$

Further from independence of increments of Brownian motion we have for B a Brownian motion independent of B_1, B_2

$$\begin{aligned}
&\mathbb{P}\left\{\exists_{t'\in[0,1]} \begin{array}{c} \forall_{t\in[t',t'+\frac{S_1}{u^2}]} W_1^*(t) > u \\ \forall_{t\in[t'+\frac{S_1}{u^2},t'+\frac{S_{1,2}}{u^2}]} W_1^*(t) > u - \frac{1}{\sqrt{u}} \end{array}\right\}\\
&\ge \mathbb{P}\left\{\exists_{t'\in[0,1]}\forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u, \forall_{s\in[0,\frac{\max(S_2-S_1,0)}{u^2}]} B(s) + c_1 s < \frac{1}{\sqrt{u}}\right\}\\
&= \mathbb{P}\left\{\exists_{t'\in[0,1]}\forall_{t\in[t',t'+\frac{S_1}{u^2}]} B_1^*(t) > u\right\} \mathbb{P}\left\{\forall_{s\in[0,\max(S_2-S_1,0)]} B(s) + \frac{c_1 s}{u} < \sqrt{u}\right\}.
\end{aligned}$$

Finally we have that

$$\lim_{u\to\infty} \mathbb{P}\left\{\forall_{s\in[0,\max(S_2-S_1,0)]} B(s) + \frac{c_1 s}{u} < \sqrt{u}\right\} = 1$$

and from Dębicki et al. (2016)[Cor 3.5] we have

$$\mathcal{P}_{[0,1],\frac{S_1}{u^2}}(c_1,u) \sim C_{\mathcal{P}}\Psi(u+c_1) \tag{3.1}$$

with $C_{\mathcal{P}} = \mathbb{E}\left\{\sup_{t\ge 0}\inf_{s\in[0,\frac{S_1}{2}]} e^{\sqrt{2}B(t-s)-2|t-s|\mathbf{1}(t>s)}\right\} \in (0,\infty)$. Further we recall that from Dębicki et al. (2021)[Thm 2.1] we have that

$$\pi_{[0,1]^2,\rho}(c_1,c_2;u,au) \sim 2\Psi(u+c_1).$$

This completes the proof of case (i).

Case (ii) : $a=\rho$. Notice that for $\Delta > 0$

$$\mathcal{P}_{S_1,S_2}(c_1,c_2;u,au) \le \mathcal{P}^*_{[1-\frac{\Delta}{u^2},1]^2,\frac{(S_1,S_2)}{u^2}}(c_1,c_2;u,au) + \pi_{[0,1]^2\setminus[1-\frac{\Delta}{u^2},1]^2,\rho}(c_1,c_2;u,au).$$

Denote $\overline{\Delta}(u) = [1-\frac{1}{\sqrt{u}},1]$. Then we have that as $u\to\infty$

$$\begin{aligned}
&\mathcal{P}^*_{[1-\frac{\Delta}{u^2},1]^2,\frac{(S_1,S_2)}{u^2}}(c_1,c_2;u,au)\\
&\le \mathbb{P}\left\{\exists_{s',t'\in\overline{\Delta}(u)}\forall_{(s,t)\in[s',s'+\frac{S_1}{u^2}]\times[t',t'+\frac{S_2}{u^2}]}W_1^*(s)>u,\, W_2^*(t)>au,\,\forall_{v\in\overline{\Delta}(u)}W_1^*(v)<u+\frac{1}{\sqrt{u}}\right\}\\
&\quad+\mathbb{P}\left\{\exists_{v\in\overline{\Delta}(u)}W_1^*(v)>u+\frac{1}{\sqrt{u}}\right\} =: \mathbb{P}_1+\mathbb{P}_2
\end{aligned}$$

Next observe that

$$\begin{aligned}
\mathbb{P}_1 &\le \mathbb{P}\left\{\exists_{s',t'\in\overline{\Delta}(u)}\forall_{(s,t)\in[s',s'+\frac{S_1}{u^2}]\times[t',t'+\frac{S_2}{u^2}]}\begin{matrix}B_1(s)-c_1s>u\\ B_2(t)-\frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}t>-\frac{\rho}{\sqrt{u}}\end{matrix},\,\forall_{v\in\overline{\Delta}(u)}B_1^*(v)<u+\frac{1}{\sqrt{u}}\right\}\\
&\le \mathbb{P}\left\{\exists_{s'\in\overline{\Delta}(u)}\forall_{s\in[s',s'+\frac{S_1}{u^2}]}B_1(s)-c_1s>u\right\}\mathbb{P}\left\{\exists_{t'\in\overline{\Delta}(u)}\forall_{t\in[t',t'+\frac{S_2}{u^2}]}B_2(t)-\frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}t>0\right\}\\
&\le \mathbb{P}\left\{\exists_{s'\in\overline{\Delta}(u)}\forall_{s\in[s',s'+\frac{S_1}{u^2}]}B_1(s)-c_1s>u\right\}\mathbb{P}\left\{\exists_{t\in\overline{\Delta}(u)}B_2(t)-\frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}t>0\right\}\\
&= \mathcal{P}^*_{[0,1],\frac{S_1}{u^2}}(c_1,u)\Phi\left(\frac{\rho c_1-c_2}{\sqrt{1-\rho^2}}\right)(1+o(1)),\quad u\to\infty.
\end{aligned}$$

Notice that with Dębicki et al. (2016) we have for some $C_1>0$

$$\lim_{u\to\infty}\frac{\mathcal{P}^*_{\overline{\Delta}(u),\frac{S_1}{u^2}}(c_1,u)}{\Psi(u+c_1)} = C_1$$

and with Dębicki and Mandjes (2015) we have for some $C_2>0$

$$\lim_{u\to\infty}\frac{\pi_{\overline{\Delta}(u),\rho}(c_1,u)}{\Psi(u+c_1)} = C_2.$$

Hence

$$\lim_{u\to\infty}\frac{\mathcal{P}^*_{\overline{\Delta}(u),\frac{S_1}{u^2}}(c_1,u)}{\pi_{\overline{\Delta}(u),\rho}(c_1,u)} = \frac{C_1}{C_2}.$$

Since from Dębicki et al. (2021) [Thm 2.1] we have

$$\mathbb{P}_2 = o\left(\pi_{\overline{\Delta}(u),\rho}(c_1,u)\right),\quad \pi_{[0,1]^2\setminus\overline{\Delta}(u)^2,\rho}(c_1,c_2;u,au) = o\left(\pi_{\overline{\Delta}(u),\rho}(c_1,u)\right),$$

hence as $u\to\infty$

$$\mathcal{P}_{S_1,S_2}(c_1,c_2;u,au) \le \Phi\left(\frac{\rho c_1-c_2}{\sqrt{1-\rho^2}}\right)\mathcal{P}^*_{[0,1],\frac{S_1}{u^2}}(c_1,u).$$

Finally, following calculations from case (i) we have that for B a Brownian motion independent of B_1, B_2

$$
\begin{aligned}
&\mathcal{P}^*_{\overline{\Delta}(u)^2, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au) \\
&\geq \mathbb{P}\left\{ \exists_{t' \in \overline{\Delta}(u)} \begin{array}{c} \forall_{t\in[t', t'+\frac{S_1}{u^2}]} B_1^*(t) > u, \sqrt{1-\rho^2}B_2(t) > (c_2 - \rho c_1)t \\ \forall_{t\in[t'+\frac{S_1}{u^2}, t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}}, \sqrt{1-\rho^2}B_2(t) > \frac{\rho}{\sqrt{u}} + (c_2-\rho c_1)t \end{array} \right\} \\
&\geq \mathbb{P}\left\{ \exists_{t' \in \overline{\Delta}(u)} \begin{array}{c} \forall_{t\in[t', t'+\frac{S_1}{u^2}]} B_1^*(t) > u \\ \forall_{t\in[t'+\frac{S_1}{u^2}, t'+\frac{S_{1,2}}{u^2}]} B_1^*(t) > u - \frac{1}{\sqrt{u}} \end{array} \right\} \\
&\quad \times \mathbb{P}\left\{ \forall_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t + \frac{\rho}{\sqrt{u}} \right\} \\
&\geq \mathbb{P}\left\{ \exists_{t'\in[0,1]} \forall_{t\in[t', t'+\frac{S_1}{u^2}]} B_1^*(t) > u \right\} \mathbb{P}\left\{ \forall_{s\in[0,\max(S_2-S_1,0)]} B(s) + \frac{c_1 s}{u} < \sqrt{u} \right\} \\
&\quad \times \mathbb{P}\left\{ \exists_{t'\in\overline{\Delta}(u)} \forall_{t\in(t', t'+\frac{S_{1,2}}{u^2})} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t + \frac{\rho}{\sqrt{u}} \right\}.
\end{aligned}
$$

Further we have

$$
\mathbb{P}\left\{ \forall_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t + \frac{\rho}{\sqrt{u}} \right\} \leq \mathbb{P}\left\{ \exists_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t \right\}.
$$

On the other hand with self-similarity and independence of increments of Brownian motion we have that for B, $\hat{B}$ Brownian motions independent of B_1, B_2

$$
\begin{aligned}
&\frac{\mathbb{P}\left\{ \forall_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t \right\}}{\mathbb{P}\left\{ \sqrt{1-\rho^2}B_2(1-\frac{1}{\sqrt{u}}) > (c_2-\rho c_1)(1-\frac{1}{\sqrt{u}}) + \frac{1}{\sqrt[8]{u}} \right\}} \\
&\geq \mathbb{P}\left\{ \forall_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t + \frac{\rho}{\sqrt{u}} \,\middle|\, \sqrt{1-\rho^2}B_2(1-\frac{1}{\sqrt{u}}) > (c_2-\rho c_1)(1-\frac{1}{\sqrt{u}}) + \frac{1}{\sqrt[8]{u}} \right\} \\
&\geq \mathbb{P}\left\{ \forall_{s\in[0,\frac{1}{\sqrt{u}}]} B(s) - (c_2-\rho c_1)s < \frac{1}{\sqrt[8]{u}} - \frac{\rho}{\sqrt{u}} \right\} = \mathbb{P}\left\{ \forall_{s\in[0,1]} \frac{1}{\sqrt[4]{u}}B(s) - \frac{1}{\sqrt{u}}(c_2-\rho c_1)s < \frac{1}{\sqrt[8]{u}} - \frac{\rho}{\sqrt{u}} \right\} \\
&= \mathbb{P}\left\{ \forall_{s\in[0,1]} B(s) - \frac{1}{\sqrt[4]{u}}(c_2-\rho c_1)s < \sqrt[4]{u} - \frac{\rho}{\sqrt[4]{u}} \right\} \sim 1
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\lim_{u\to\infty} \mathbb{P}\left\{ \exists_{t\in\overline{\Delta}(u)} \sqrt{1-\rho^2}B_2(t) > (c_2-\rho c_1)t \right\} \\
&= \lim_{u\to\infty} \mathbb{P}\left\{ \sqrt{1-\rho^2}B_2(1-\frac{1}{\sqrt{u}}) > (c_2-\rho c_1)(1-\frac{1}{\sqrt{u}}) + \frac{1}{\sqrt[8]{u}} \right\} = \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right).
\end{aligned}
$$

Hence the claim follows from (3.1) and from Dębicki et al. (2021)[Thm 2.1], which gives

$$
\pi_{[0,1]^2,\rho}(c_1, c_2; u, au) \sim 2\Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right)\Psi(u+c_1). \quad \square
$$

3.2. Proof of Theorem 2.2

Before we begin the proof we need few technical lemmas. First let

$$
\Sigma_{s,t} = \begin{pmatrix} s & \rho\min(s,t) \\ \rho\min(s,t) & t \end{pmatrix}
$$

be the covariance matrix of $(W_1(s), W_2(t))$. In Dębicki et al. (2021) it was noted that the drift has a significant impact on the optimization problem that was used to determine asymptotics for the classical ruin. We denote below for $\boldsymbol{a} = (1+\frac{c_1 s}{u}, a + \frac{c_2 t}{u})^\top$

$$
q_{\boldsymbol{a}}(s,t) := \boldsymbol{a}^\top \Sigma_{s,t}^{-1}\boldsymbol{a}, \quad \boldsymbol{b}(s,t) := \Sigma_{s,t}^{-1}\boldsymbol{a}
$$

and set

$$q^*_{\boldsymbol{a}}(s,t)=\min_{\boldsymbol{x}\geq\boldsymbol{a}} q_{\boldsymbol{x}}(s,t),\quad q^*_{\boldsymbol{a}}=\min_{s,t\in[0,1]} q^*_{\boldsymbol{a}}(s,t). \tag{3.2}$$

Note that for $a>\rho$ and large enough u we have $\boldsymbol{b}(s,t)\sim(\frac{t-a\rho\min(s,t)}{st-\rho^2(\min(s,t))^2},\frac{as-\rho\min(s,t)}{st-\rho^2(\min(s,t))^2})>\boldsymbol{0}$. From Dębicki et al. (2010) we have that for any s,t positive the following logarithm asymptotics occurs

$$\lim_{u\to\infty}\frac{1}{u^2}\log\mathbb{P}\left\{\exists_{s,t\in[0,1]}W_1^*(s)>u,W_2^*(t)>au\right\}=-\frac{q^*_{\boldsymbol{a}}}{2}. \tag{3.3}$$

Hence we will use the function $q^*_{\boldsymbol{a}}(s,t)$ to reflect the asymptotics of $\mathbb{P}\left\{W_1^*(s)>u,W_2^*(t)>au\right\}$. Recall the optimization results first calculated in Dębicki et al. (2021)[Lemma 3.1]

Lemma 3.1. *For all large u we have:*

(i) If $a=1,\rho\in(-1,-\frac{1}{2})$, then $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ attains its unique local minima on $[0,1]^2$ at

$$(s_u,t_u):=\left(1,\frac{1}{\rho(2\rho-1)+\frac{c_2-\rho c_1}{u}}\right),\quad(\bar{s}_u,\bar{t}_u):=\left(\frac{1}{\rho(2\rho-1)+\frac{c_1-\rho c_2}{u}},1\right).$$

(ii) If $a=1,\rho=-\frac{1}{2}$, then $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ attains its unique local minima on $[0,1]^2$ at

$$(s_u,t_u):=\left(1,\min(\frac{1}{1+\frac{c_2+2c_1}{u}},1)\right),\quad(\bar{s}_u,\bar{t}_u):=\left(\min(\frac{1}{1+\frac{c_1+2c_2}{u}},1),1\right).$$

(iii) For any other $a\in(\max(0,\rho),1]$, $\rho\in(-1,1)$, $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ attains its unique minimum on $[0,1]^2$ at

$$(s_u,t_u):=\begin{cases}(1,\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}), & \textit{if } \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\in[0,1]\\ (1,1), & \textit{otherwise}.\end{cases}$$

In the rest of the paper we denote

$$t^*:=\lim_{u\to\infty}t_u.$$

Further let $k_u=1-\frac{(k-1)\Delta}{u^2}$, $l_u=t_u-\frac{(l-1)\Delta}{u^2}$, $u>0$, $\Delta>0$ and set

$$E_{u,k}=[(k+1)_u,k_u],\ E_{u,k,l}=E_{u,k}\times E_{u,l},\quad E=[-\Delta,0]\times[-\Delta,0].$$

Define also $\eta_{u,k,l}(s,t):=(\eta_{1,u,k}(s),\eta_{2,u,l}(t)):=u(W_1(\frac{s}{u^2}+k_u)-W_1(k_u)-c_1\frac{s}{u^2},W_2(\frac{t}{u^2}+l_u)-W_2(l_u)-c_2\frac{t}{u^2})$. The following lemma is used to calculate the ruin probability on an interval of size of order $O(\frac{1}{u^2})$.

Lemma 3.2. *Let $\rho\in(-1,1)$, $a\in(\max(0,\rho),1]$, $l,k=O(\frac{u\log(u)}{\Delta})$ and $\Delta,S_1,S_2>0$ be given constants. Then, as $u\to\infty$*

$$\mathcal{P}^*_{E_{u,k,l},\frac{(S_1,S_2)}{u^2}}(c_1,c_2,u,au)\sim u^{-2}\varphi_t^*(u+c_1,au+c_2t_u)I_1(\Delta)e^{-\frac{1}{2}u^2(q_a^*(k_u,l_u)-q_a^*(1,t_u))},$$

$$I_1(\Delta)=\begin{cases}\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{s',t'\in[0,\Delta]}\forall_{s\in[s',s'+S_1],t\in[t',t'+S_2]}:\begin{array}{c}W_1(s)-s>x\\ W_2(t)-at>y\end{array}\right\}e^{\lambda_1x+\lambda_2y}dxdy & l_u=k_u\\ \displaystyle\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{t'\in[0,\Delta]}\forall_{t\in[t',t'+S_2]}:W_2(t)-\lambda_2t>y\right\} & l_u>k_u\\ \times\mathbb{P}\left\{\exists_{s'\in[0,\Delta]}\forall_{s\in[s',s'+S_1]}:W_1(s)-s>x\right\}e^{\lambda_1x+\lambda_2y}dxdy & \\ \displaystyle\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{s'\in[0,\Delta]}\forall_{s\in[s',s'+S_1]}:W_1(s)-\lambda_1s>x\right\} & l_u<k_u,\\ \times\mathbb{P}\left\{\exists_{t'\in[0,\Delta]}\forall_{t\in[t',t'+S_2]}:W_2(t)-\frac{a}{t^*}t>y\right\}e^{\lambda_1x+\lambda_2y} & \end{cases}$$

and $$\lambda_1=\begin{cases}\frac{1}{t^*}\frac{1-a\rho}{1-\rho^2} & l_u=k_u\\ \frac{t^*-a\rho}{t^*-\rho^2}, & l_u>k_u\\ \frac{1-a\rho}{1-\rho^2t^*}, & l_u<k_u\end{cases},\quad\lambda_2=\begin{cases}\frac{1}{t^*}\frac{a-\rho}{1-\rho^2}, & l_u=k_u\\ \frac{a-\rho}{t^*-\rho^2}, & l_u>k_u\\ \frac{a-\rho t^*}{t^*-\rho^2(t^*)^2}, & l_u<k_u.\end{cases}$$

Additionally

$$\lim_{u\to\infty}\sup_{l,k=O(u\log u)}\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{(s',t')\in E}\forall_{s\in[s',s'+S_1],t\in[t',t'+S_2]}:\ \eta_{u,k,l}(s,t)>(x,y)\ \middle|\ \begin{array}{c}W_1^*(k_u)=u-\frac{x}{u}\\ W_2^*(l_u)=au-\frac{y}{u}\end{array}\right\}e^{\lambda_1x+\lambda_2y}dxdy<\infty. \tag{3.4}$$

The proof of the above lemma is identical to the proof of Dębicki et al. (2021)[Lemma 3.4] for a different functional, which does not influence the technique and hence the proof is omitted.

Proof of Theorem 2.2. We split the main part of the proof into several cases which depend on the behaviour of the variance and the optimization point we get from Dębicki et al. (2021)[Lem. 3.1]. The proofs for cases (iii)–(v) are very similar and analogous to the proof of case (ii) and proofs from Dębicki et al. (2021)[Thm 2.2] and hence are omitted. Let next

$$N_u := \lfloor \frac{u\log(u)}{\Delta} \rfloor, \quad K_u^{(1)} = \frac{(c_2 - c_1\rho)u}{\Delta}, \quad K_u^{(2)} = \frac{(c_1 - c_2\rho)u}{\Delta},$$

$$E^1_{u,m} := [(m+1)_u, m_u], \quad E^2_{u,j} := [(j+1)_u, j_u],$$

where $m_u = 1 - \frac{(m-1)\Delta}{u^2}, j_u = t_u - \frac{(j-1)\Delta}{u^2}$. For $\Delta > 0$ we have for any $F_u \subset [0,1]^2$

$$\mathscr{P}_{S_1,S_2}(c_1, c_2; u, au) \geq \frac{\mathcal{P}^*_{F_u, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au)}{\pi_{[0,1]^2}(c_1, c_2; u, au)}.$$

On the other hand

$$\mathscr{P}_{S_1,S_2}(c_1, c_2; u, au) \leq \frac{\mathcal{P}^*_{F_u, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au) + \pi_{[0,1]^2\setminus F_u}(c_1, c_2; u, au)}{\pi_{[0,1]^2}(c_1, c_2; u, au)}.$$

Since from Dębicki et al. (2021)[Thm 2.2] we have that

$$\lim_{u\to\infty} \frac{\pi_{[0,1]^2\setminus F_u}(c_1, c_2; u, au)}{\pi_{[0,1]^2}(c_1, c_2; u, au)} = 0$$

therefore

$$\mathscr{P}_{S_1,S_2}(c_1, c_2; u, au) \sim \frac{\mathcal{P}^*_{F_u, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au)}{\pi_{[0,1]^2}(c_1, c_2; u, au)},$$

where F_u is case dependent.

Case (i) : $\rho > \frac{1}{4a}(1 - \sqrt{8a^2+1})$. According to Dębicki et al. (2021)[Lem. 3.1] $t^* = t_u = 1$. From Dębicki et al. (2021)[Thm 2.2, case (i)] we have $F_u := E^2_{u,1}$. Using Lemma 3.2 and taking $u \to \infty$ and then $\Delta \to \infty$, we get that

$$\mathcal{P}^*_{E^2_{u,1}, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au) \sim Iu^{-2}\varphi_1(u + c_1, au + c_2),$$

where $I = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{s',t'\in[0,\infty)}\forall_{s\in[s',s'+S_1],t\in[t',t'+S_2]} : \begin{array}{c} W_1(s) - s > x \\ W_2(t) - at > y \end{array}\right\} e^{\lambda_1 x + \lambda_2 y}dxdy$. Positivity of I comes of the constant from the fact that the function that we integrate is positive on a set of positive mass. Finiteness follows straightforwardly from

$$\mathbb{P}\left\{\exists_{s',t'\in\mathbb{R}_+}\forall_{s\in[s',s'+S_1],t\in[t',t'+S_2]} : \begin{array}{c} W_1(s) - s > x \\ W_2(t) - at > y \end{array}\right\} \leq \mathbb{P}\left\{\exists_{s',t'\in\mathbb{R}_+} : \begin{array}{c} W_1(s') - s' > x \\ W_2(t') - at' > y \end{array}\right\}$$

and Dębicki et al. (2021) [Lemma 3.6]. With that, the proof of case (i) is complete.

Case (ii) : $\rho = \frac{1}{4a}(1 - \sqrt{8a^2+1})$. We split this case into two subcases since the behaviour of the optimizing point is slightly different. First let, $c_2 - \rho c_1 \leq 0$. According to Dębicki et al. (2021)[Lem. 3.1] $t^* = t_u = 1$. From Dębicki et al. (2021)[Thm 2.2, case (ii)] we have $F_u := [1 - \frac{\Delta}{u^2}, 1] \times [1 - \frac{\log(u)}{u}, 1 - \frac{\Delta}{u^2}]$. Using Bonferroni inequality we have that

$$\begin{aligned}
&\mathcal{P}^*_{F_u, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au) \\
&\geq \sum_{l=2}^{N_u} \mathbb{P}\left\{\exists_{s'\in E^1_{u,1}, t'\in E^2_{u,l}}\forall_{s\in[s',s'+\frac{S_1}{u^2}]}, t \in [t', t' + \frac{S_2}{u^2}] : W_1^*(s) > u, W_2^*(t) > au\right\} \\
&\quad - \sum_{l=2}^{N_u}\sum_{m=l+1}^{N_u} \mathbb{P}\left\{\exists_{s\in E^1_{u,1}, t_1\in E^2_{u,l}, t_2\in E^2_{u,m}} : W_1^*(s) > u, W_2^*(t_1) > au, W_2^*(t_2) > au\right\} \\
&:= P_{u,\Delta} - D_{u,\Delta}.
\end{aligned} \tag{3.5}$$

Further we have

$$\mathcal{P}^*_{F_u, \frac{(S_1,S_2)}{u^2}}(c_1, c_2; u, au) \leq P_{u,\Delta} + D_{u,\Delta}. \tag{3.6}$$

From Lemma 3.2 we have as $u \to \infty$

$$P_{u,\Delta} \sim C^{(1)}_{2,\mathcal{P}}(\Delta)C^{(2)}_{2,\mathcal{P}}(\Delta)u^{-2}\varphi^*_t(u+c_1, au+c_2)\sum_{l=2}^{N_u} e^{-\frac{1}{2}u^2(q_{\mathbf{a}}(k_u,l_u)-q_{\mathbf{a}}(1,1))},$$

where

$$C^{(1)}_{2,\mathcal{P}}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s'\in[0,\Delta]}\forall_{s\in[s',s'+S_1]} : W_1(s) - \frac{1-a\rho}{1-\rho^2}s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x}dx$$

and

$$C^{(2)}_{2,\mathcal{P}}(\Delta) = \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t'\in[0,\Delta]}\forall_{t\in[t',t'+S_2]} : W_2(t) - at > x\right\} e^{2ax}dx.$$

Using Taylor expansion we have that for $k < l$

$$u^2(q_a(k_u, l_u) - q_a(1,1)) = \tau_1(k-1)\Delta + \tau_4\frac{(l-1)^2\Delta^2}{u^2} + o(\frac{k^2}{u^2}) + o(\frac{l^3}{u^4}),$$

where $\tau_1 = \frac{(1-a\rho)^2}{(1-\rho^2)^2} > 0$ and $\tau_4 = \frac{\rho^2-2a\rho^3+a^2\rho^2}{(1-\rho^2)^2} > 0$. Therefore with Dębicki et al. (2021)[Lem 3.6] we have as $u \to \infty$

$$\begin{aligned} P_{u,\Delta} &\sim C^{(1)}_{2,\mathcal{P}}(\Delta)C^{(2)}_{2,\mathcal{P}}(\Delta)u^{-2}\varphi^*_t(u+c_1, au+c_2)\sum_{l=2}^{N_u} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}} \\ &= \frac{1}{\sqrt{\tau_4}}C^{(1)}_{2,\mathcal{P}}(\Delta)\frac{C^{(2)}_{2,\mathcal{P}}(\Delta)}{\Delta}u^{-1}\varphi^*_t(u+c_1, au+c_2)\sum_{l=2}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}} \\ &\sim C^{(1)}_{2,\mathcal{P}}(\Delta)\frac{C^{(2)}_{2,\mathcal{P}}(\Delta)}{\Delta}\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi^*_t(u+c_1, au+c_2). \end{aligned}$$

From Dębicki et al. (2016)(8) we have that $\lim_{\Delta\to\infty} C^{(1)}_{2,\mathcal{P}}(\Delta) = C^{(1)}_{2,\mathcal{P}} \in (0,\infty)$ and from Dębicki et al. (2015)(2.5) $\lim_{\Delta\to\infty}\frac{C^{(2)}_{2,\mathcal{P}}(\Delta)}{\Delta} = C^{(2)}_{2,\mathcal{P}} \in (0,\infty)$. Hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{P_{u,\Delta}}{C^{(1)}_{2,\mathcal{P}}C^{(2)}_{2,\mathcal{P}}\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi^*_t(u+c_1, au+c_2)} = 1.$$

From the proof of Dębicki et al. (2021)[Theorem 2.2, case (ii)] we have that for $C = C^{(1)}_{2,\mathcal{P}}C^{(2)}_{2,\mathcal{P}}\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}$

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{D_{u,\Delta}}{P_{u,\Delta}} = \lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{D_{u,\Delta}}{Cu^{-1}\varphi_{t^*}(u+c_1, au+c_2)} = 0. \tag{3.7}$$

i.e. the double sum is negligible compared to the single sum.

Now we consider the case of $c_2 - \rho c_1 > 0$. According to Dębicki et al. (2021)[Lem. 3.1] there is exactly one minimizer of $q^*_{\mathbf{a}_u(s,t)}(s,t)$ on $[0,1]^2$: $(s_u, t_u) = \left(1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\right)$, where from $c_2 - \rho c_1 > 0$ we obtain that

$$\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}} \nearrow 1$$

as $u \to \infty$. From Dębicki et al. (2021)[Thm 2.2, case (iii)] we have $F_u := [1-\frac{\Delta}{u^2}, 1] \times [t_u - \frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}]$ and since the proof of this part is vastly similar to the previous one, we omit it.

Case (iii) : $\rho = -\frac{1}{2}, a = 1$. According to Dębicki et al. (2021)[Lem. 3.1] $t^* = 1$. The proof is analogous to case (ii). We use (3.6) and (3.5) with

$$F_u := [1-\frac{\Delta}{u^2}, 1] \times [1-\frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}] \cup [1-\frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}] \times [1-\frac{\Delta}{u^2}, 1].$$

Case (iv) : $\rho < \frac{1}{4a}(1-\sqrt{8a^2+1})$. From Dębicki et al. (2021)[Lem. 3.1] we have exactly one minimizer of $q^*_{\mathbf{a}_u(s,t)}(s,t)$ on $[0,1]^2$ which is $(s_u, t_u) = (1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}})$ and for large enough u we have $t_u < 1$. The proof is analogous to case (ii) with

$$F_u := [1-\frac{\Delta}{u^2}, 1] \times [t_u - \frac{\log(u)}{u}, t_u + \frac{\log(u)}{u}].$$

Case (v) : $a = 1$, $\rho < A_a$. According to Dębicki et al. (2021)[Lem. 3.1], there are two optimal points:

$$(s_u, t_u) = (1, \frac{1}{\rho(2\rho - 1) + \frac{c_2 - \rho c_1}{u}}), \quad (\bar{s}_u, \bar{t}_u) = (\frac{1}{\rho(2\rho - 1) + \frac{c_1 - \rho c_2}{u}}, 1),$$
$$F_u := [1 - \frac{\Delta}{u^2}, 1] \times [t_u - \frac{\log(u)}{u}, t_u + \frac{\log(u)}{u}] \cup [t_u - \frac{\log(u)}{u}, t_u + \frac{\log(u)}{u}] \times [1 - \frac{\Delta}{u^2}, 1]. \quad \square$$

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References

Bai, L., Luo, L., 2017. Parisian ruin of the Brownian motion risk model with constant force of interest. Statist. Probab. Lett. 120, 34–44.
Czarna, I., Renaud, J.-F., 2016. A note on Parisian ruin with an ultimate bankruptcy level for Lévy insurance risk processes. Statist. Probab. Lett. 113, 54–61.
Dassios, A., Wu, S., 2008. Parisian ruin with exponential claims.
Dassios, A., Wu, S., 2011. Semi-Markov model for excursions and occupation time of Markov processes.
Dębicki, K., Hashorva, E., Ji, L., 2015. Parisian ruin of self-similar Gaussian risk processes. J. Appl. Probab. 52 (3), 688–702.
Dębicki, K., Hashorva, E., Ji, L., 2016. Parisian ruin over a finite-time horizon. Sci. China Math. 59 (3), 557–572.
Dębicki, K., Hashorva, E., Krystecki, K., 2021. Finite-time ruin probability for correlated Brownian motions. Scand. Actuar. J. 1–26.
Dębicki, K., Hashorva, E., Michna, Z., 2020. Simultaneous ruin probability for two-dimensional Brownian risk model. J. Appl. Probab. 57 (2), 597–612.
Dębicki, K., Kosiński, K., Mandjes, M., Rolski, T., 2010. Extremes of multidimensional Gaussian processes. Stochastic Process. Appl. 120 (12), 2289–2301.
Dębicki, K., Mandjes, M., 2015. Queues and Lévy Fluctuation Theory. Springer International Publishing.
Delsing, G., Mandjes, M., Spreij, P., Winands, E., 2018. Asymptotics and approximations of ruin probabilities for multivariate risk processes in a Markovian environment. arXiv preprint arXiv:1812.09069.
Kriukov, N., 2020. Parisian & cumulative Parisian ruin probability for two-dimensional Brownian risk model. arXiv preprint arXiv:2001.09302.
Loeffen, R., Czarna, I., Palmowski, Z., 2013. Parisian ruin probability for spectrally negative Lévy processes. Bernoulli 19, 599–609.