FINITE-TIME RUIN PROBABILITY FOR CORRELATED BROWNIAN MOTIONS

KRZYSZTOF DĘBICKI, ENKELEJD HASHORVA, AND KONRAD KRYSTECKI

Abstract: Let $(W_1(s), W_2(t)), s,t \ge 0$ be a two dimensional Gaussian process with standard Brownian motion marginals and constant correlation $\rho \in (-1,1)$. Define the joint survival probability of both supremum functionals by

$$\pi_\rho(c_1, c_2; u, v) = \mathbb{P}\left\{ \sup_{s\in[0,1]} (W_1(s) - c_1 s) > u, \sup_{t\in[0,1]} (W_2(t) - c_2 t) > v \right\},$$

where $c_1, c_2 \in \mathbb{R}$ and u, v are given positive constants. Approximation of $\pi_\rho(c_1, c_2; u, v)$ is of interest for the analysis of ruin probability in bivariate Brownian risk model as well as in the study of power of bivariate test statistics. In this contribution we derive tight bounds for $\pi_\rho(c_1, c_2; u, v)$ in the case $\rho \in (0,1)$ and obtain precise approximations for all $\rho \in (-1,1)$ by letting $u \to \infty$ and taking $v = au$ for some fixed positive constant a.

Key Words: Two-dimensional Brownian motion; Exact asymptotics; Ruin probability

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Consider the bivariate Brownian risk model $(R_1(s), R_2(t)), s,t \ge 0$ of two insurance risk portfolios

$$R_1(s) = u + c_1 s - W_1(s), \quad R_2(t) = v + c_2 t - W_2(t), \quad s,t \ge 0,$$

where the random process of accumulated claims $(W_1(s), W_2(t))\,, s,t \ge 0$ is assumed to be jointly Gaussian, the initial capitals are u, v and the corresponding premium rates are c_1, c_2. The covariance structure of the bivariate process (W_1, W_2) can be quite general. In view of e.g., [1] (see also [2]) a natural choice for the bivariate risk model is to suppose that marginally $W_i's$ are standard Brownian motions with constant correlation $\rho \in (-1,1)$, i.e.,

$$(W_1(s), W_2(t)) = (B_1(s), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)), \quad s,t \ge 0, \tag{1.1}$$

where B_1, B_2 are two independent standard Brownian motions.

The ruin probability of a single portfolio in the time horizon $[0,T], T > 0$ is given by (see e.g., [3])

$$\pi_T^{(1)}(c_i; u) = \mathbb{P}\left\{ \inf_{t\in[0,T]} R_i(t) < 0 \right\} = \mathbb{P}\left\{ \sup_{t\in[0,T]} (W_i(t) - c_i t) > u \right\}$$

Date: March 13, 2021.

$$= \Phi\left(-\frac{u}{\sqrt{T}} - c_i\sqrt{T}\right) + e^{-2c_i u}\Phi\left(-\frac{u}{\sqrt{T}} + c_i\sqrt{T}\right) \tag{1.2}$$

for $i = 1, 2$ and all $u \geq 0$; here we set $\Phi(x) = 1 - \Psi(x) = \mathbb{P}\{B_1(1) \leq x\}$.

Define next the component-wise ruin probability on $[0, T]$ by

$$\widetilde{\pi}_{T,\rho}(c_1, c_2; u, v) = \mathbb{P}\left\{\inf_{s\in[0,T]} R_1(s) < 0, \inf_{t\in[0,T]} R_2(t) < 0\right\} = \mathbb{P}\left\{\sup_{s\in[0,T]} W_1^*(s) > u, \sup_{t\in[0,T]} W_2^*(t) > v\right\},$$

where $W_i^*(s) = W_i(s) - c_i s$. By the self-similarity of Brownian motion, without loss of generality we shall suppose that $T = 1$ and set

$$\pi_\rho(c_1, c_2; u, v) := \widetilde{\pi}_{1,\rho}(c_1, c_2; u, v).$$

$\widetilde{\pi}_{T,\rho}$ has been investigated in [4–7]. In particular, when $\rho \neq 0$, [5][Thm 2.2] derives a formula for

$$\mathbb{P}\left\{\sup_{s\in[0,T]} W_1^*(s) \leq u, \sup_{t\in[0,T]} W_2^*(t) \leq v\right\},$$

which is given in terms of infinite-series and Bessel functions. Those representations are complex and do not allow to observe the behaviour of $\widetilde{\pi}_{T,\rho}$ for large u, v. The infinite-time horizon counterpart of $\widetilde{\pi}_{T,\rho}$ is studied in [8, 9], where both logarithmic and exact asymptotics for $\widetilde{\pi}_{\infty,\rho}(c_1, c_2; u, u)$, as $u \to \infty$ is derived. The results of the aforementioned paper differ from our findings here, since we consider a finite time horizon.

In [2] the simultaneous ruin probability

$$\overline{\pi}_\rho(c_1, c_2; u, au) = \mathbb{P}\{\exists s \in [0, 1] : W_1^*(s) > u, W_2^*(s) > au\}, \quad a \leq 1$$

has been studied. Therein an upper bound for $\overline{\pi}_\rho$ is derived in terms of the following joint survival function

$$p_{u,v} := \mathbb{P}\{W_1^*(1) > u, W_2^*(1) > v\}.$$

It turns out that an accurate upper bound can also be derived for π_ρ if $\rho \in (0, 1)$.

Theorem 1.1. *If $\rho \in (0, 1)$, then for all $u, v \geq 0$*

$$p_{u,v} \leq \pi_\rho(c_1, c_2; u, v) \leq A(c_1, c_2) p_{u,v}, \tag{1.3}$$

where $1/A(c_1, c_2) = \Psi\big(\max(0, \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}})\big)\Psi(\max(0, c_1))$.

The upper bound in (1.3) is given in terms of $p_{u,v}$ and the constant $A(c_1, c_2)$, which does not depend on u and v. This suggests that asymptotically, as $u \to \infty$

$$\pi_\rho(c_1, c_2; u, au) \sim C p_{u,au}, \tag{1.4}$$

where $C > 0$ is some constant and $f \sim g$ means $\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1$. Such asymptotic behaviour is already observed for the probability of simultaneous ruin in [2]. As we shall show in the next section, which contains the main results of this paper, this statement does not apply for all $\rho \in (-1, 0)$.

Brief outline of the rest of the paper. In Section 2 we present our main findings. Theorems 2.1 treats relatively simple cases where the conjecture in (1.4) can be confirmed. All other cases, which are more complex are treated in Theorem 2.2 . In Section 3 we present the proofs of the main results along with some additional findings, whereas in Appendix we collect the proofs of several technical results.

2. Main Results

Below c_1, c_2 are given constants and without loss of generality we suppose that $a \in (0, 1]$. Recall that $W_i^*(t) = W_i(t) - c_i t, t \geq 0$. Our results in this section are motivated by the conjecture in (1.4). It can be confirmed in several cases, however a detailed and complex analysis shows that other cases also occur.

Our investigation shall focused on two scenarios:

(i) case $1 > \rho \geq a > 0$, when one coordinate asymptotically dominates the other, leading to the reduction of dimension phenomena,

and (ii) - the remaining cases, where both coordinates contribute to the asymptotics.

2.1. **Dimension-reduction case.** Suppose that $1 > \rho \geq a > 0$. It turns out that for this scenario conjecture (1.4) is true. We have the following result:

Theorem 2.1. *(i) If $1 > \rho > a > 0$, then $\pi_\rho(c_1, c_2; u, au) \sim \pi_1^{(1)}(c_1; u) \sim 2\mathbb{P}\{W_1^*(1) > u\}$.*
(ii) If $\rho = a \in (0, 1)$, then $\pi_\rho(c_1, c_2; u, au) \sim \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right) \pi_1^{(1)}(c_1; u)$.

2.2. **Full-dimensional case.** Consider next the complementary scenario to the dimension-reduction case, i.e., $\rho \in (-1, 1)$ and $a \in (\max(0, \rho), 1]$. This case requires a more detailed analysis divided on several subcases, which need separate treatment leading to six different asymptotic regimes.

Before presenting the main result of this section we introduce some useful notation. Let $\varphi_{s,t}$ be the probability density function (pdf) of $(W_1(s), W_2(t))$ and let $\varphi_t := \varphi_{1,t}$. Next Σ_X denotes the covariance matrix of a random vector X and set for $\rho \in (-1, 1)$

$$\Sigma_{s,t} := \Sigma_{(W_1(s), W_2(t))} = \begin{pmatrix} s & \rho \min(s,t) \\ \rho \min(s,t) & t \end{pmatrix}, s, t \geq 0. \tag{2.1}$$

Define further two important constants

$$A_a = \frac{1}{4a}(1 - \sqrt{8a^2 + 1}), \quad t_* = \frac{a}{\rho(2a\rho - 1)}, \ a > 0.$$

Theorem 2.2. *Consider $\rho \in (-1, 1)$ and $a \in (\max(0, \rho), 1]$.*

(i) If $\rho > A_a$, then

$$\pi_\rho(c_1, c_2; u, au) \sim C_1 u^{-2} \varphi_1(u + c_1, au + c_2),$$

where

$$C_1 = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{s,t\in[0,\infty)} : \begin{array}{c} W_1(s) - s > x \\ W_2(t) - at > y \end{array} \right\} e^{\frac{1-a\rho}{1-\rho^2}x + \frac{a-\rho}{1-\rho^2}y} dxdy \in (0,\infty).$$

(ii) If $\rho = A_a$ and $a < 1, c_2 - c_1\rho \le 0$, then

$$\pi_\rho(c_1, c_2; u, au) \sim C_2 u^{-1} \varphi_1(u + c_1, au + c_2),$$

where $C_2 = \frac{a(1-\rho^2)}{1-a\rho}\frac{\sqrt{2\pi}}{\sqrt{\tau}}, \tau = \frac{\rho^2 - 2a\rho^3 + a^2\rho^2}{(1-\rho^2)^2} > 0.$

(iii) If $\rho = A_a$ and $a < 1, c_2 - c_1\rho > 0$, then

$$\pi_\rho(c_1, c_2; u, au) \sim C_3 u^{-1} \varphi_1(u + c_1, au + c_2),$$

where $C_3 = \frac{2a(1-\rho^2)}{1-a\rho} e^{-a\frac{(c_1\rho - c_2)^2}{2\rho(1-a\rho)}} \frac{\sqrt{2\pi}}{\sqrt{\tau}} \Phi\left(c_2 - \rho c_1\right), \tau = \frac{\rho^2 - 2a\rho^3 + a^2\rho^2}{(1-\rho^2)^2} > 0.$

(iv) If $\rho = A_a, a = 1$, then

$$\pi_\rho(c_1, c_2; u, u) \sim C_4 u^{-1} \varphi_1(u + c_1, u + c_2),$$

where

$$\sqrt{\frac{2}{3\pi}} C_4 = \begin{cases} e^{-2\frac{(\frac{1}{2}c_1 + c_2)^2}{3}} \Phi\left(c_2 + \frac{1}{2}c_1\right) + e^{-2\frac{(\frac{1}{2}c_2 + c_1)^2}{3}} \Phi\left(c_1 + \frac{1}{2}c_2\right), & c_2 > \max(-\frac{1}{2}c_1, -2c_1) \\ e^{-2\frac{(\frac{1}{2}c_1 + c_2)^2}{3}} \Phi\left(c_2 + \frac{1}{2}c_1\right) + \frac{1}{2}, & -\frac{1}{2}c_1 < c_2 \le -2c_1 \\ \frac{1}{2} + e^{-2\frac{(\frac{1}{2}c_2 + c_1)^2}{3}} \Phi\left(c_1 + \frac{1}{2}c_2\right), & -2c_1 < c_2 \le -\frac{1}{2}c_1 \\ 1, & c_2 \le \min(-\frac{1}{2}c_1, -2c_1). \end{cases}$$

(v) If $\rho < A_a$ and $a < 1$, then

$$\pi_\rho(c_1, c_2; u, au) \sim C_5 u^{-1} \varphi_{t_*}(u + c_1, au + c_2 t_*),$$

where $C_5 = e^{-a\frac{(c_1\rho - c_2)^2}{2\rho(1-a\rho)}} \frac{2a\sqrt{2\pi}}{t_*\sqrt{\tau}} \frac{1}{1-2a\rho}, \tau = -\frac{\rho^3(1-2a\rho)^4}{a(1-a\rho)} > 0.$

(vi) If $\rho < A_a$ and $a = 1$, then

$$\pi_\rho(c_1, c_2; u, u) \sim C_6 u^{-1} \varphi_{t_*}(u + \min(c_1, c_2), u + \max(c_1, c_2) t_*),$$

where $\tau = -\frac{\rho^3(1-2\rho)^4}{1-\rho} > 0$ *and*

$$C_6 = \begin{cases} e^{-\frac{(\min(c_1,c_2)\rho - \max(c_1,c_2))^2}{2\rho(1-\rho)}} \frac{2\sqrt{2\pi}}{t_*\sqrt{\tau}} \frac{1}{1-2\rho}, & c_1 \ne c_2 \\ 2e^{-\frac{c_2^2(1-\rho)}{2\rho}} \frac{2\sqrt{2\pi}}{t_*\sqrt{\tau}} \frac{1}{1-2\rho}, & c_1 = c_2. \end{cases}$$

In case (i) we still have that (1.4) holds. However the claim of Theorem 1.1 (and (1.4)) is not true for $\rho \in (-1, A_a]$ and $a \in (0,1]$. This is related to the fact that when $\rho < 0$ is relatively large compared to a (in terms of absolute value), then it is less likely that the ruin occurs simultaneously. Hence the region that determines the asymptotics is separated from point $(1,1)$ and the ruin is truly non-simultaneous. In those cases we can observe a different behaviour to that conjectured in (1.4) (this means that the bounds in (1.3) do not hold for those cases), namely

$$\pi_\rho(c_1, c_2; u, au) \sim C u p_{u,au}, \quad u \to \infty \tag{2.2}$$

for some $C \in (0,\infty)$. A special case is if $a = 1$ for which $A_a = -1/2$. Then Case (i),(iv),(vi) in the above theorem are possible.

3. Proofs

3.1. **Proof of Theorem 1.1.** Given two independent standard Brownian motions B_1, B_2 let

$$S_1 := \sup_{t\in[0,1]} (B_1(t) - c_1 t), \quad S_2 := \sup_{t\in[0,1]} \left(\sqrt{1-\rho^2} B_2(t) - (c_2 - \rho c_1)t\right).$$

Additionally, let $g_1(\cdot), g_2(\cdot)$ be probability density functions of S_1 and $\mathcal{X} := \sqrt{1-\rho^2}B_2(1) - (c_2 - \rho c_1)$, respectively. Since $W_2(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t)$, we have for all $u, v \ge 0$

$$\pi_\rho(c_1, c_2; u, v) \le \mathbb{P}\{S_1 > u, \rho S_1 + S_2 > v\}. \tag{3.1}$$

For all $c \in \mathbb{R}$ and all $u \ge 0$

$$\mathbb{P}\left\{\sup_{t\in[0,1]} (B(t) - ct) > u\right\} \le \frac{\mathbb{P}\{B(1) > u + c\}}{\Psi\left(\max(0, c)\right)}$$

see e.g., [2] and [10][Thm 1.1]. Hence setting $1/A(c_1, c_2) = \Psi(\max(0, \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}))\Psi(\max(0, c_1))$ we have

$$\begin{aligned}
&\mathbb{P}\{S_1 > u, \rho S_1 + S_2 > v\}\\
&= \int_u^\infty \mathbb{P}\{S_2 > v - \rho x\} g_1(x) dx\\
&\le \frac{1}{\Psi\left(\max\left(0, \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}\right)\right)} \int_u^\infty \mathbb{P}\{\mathcal{X} > v - \rho x\} g_1(x) dx\\
&= \frac{1}{\Psi\left(\max\left(0, \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}\right)\right)} \mathbb{P}\{S_1 > u, \rho S_1 + \mathcal{X} > v\}\\
&= \frac{1}{\Psi\left(\max\left(0, \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}}\right)\right)} \int_{-\infty}^\infty \mathbb{P}\left\{S_1 > u, S_1 > \frac{v-x}{\rho}\right\} g_2(x) dx\\
&\le A(c_1, c_2) \int_{-\infty}^\infty \mathbb{P}\{B_1(1) - c_1 > u, \rho(B_1(1) - c_1) > v - x\} g_2(x) dx
\end{aligned}$$

$$= \quad A(c_1,c_2)\mathbb{P}\{W_1(1) > u + c_1, W_2(1) > v + c_2\}$$

establishing the proof. □

3.2. **Proof of Theorem 2.1.** First note that for all $u > 0$

$$\pi_\rho(c_1,c_2;u,au) \quad \geq \quad \mathbb{P}\{\exists t \in [0,1] : W_1^*(t) > u, W_2^*(t) > au\}. \tag{3.2}$$

Case $0 < a < \rho < 1$. Next, for all $u > 0$

$$\pi_\rho(c_1,c_2;u,au) \leq \mathbb{P}\left\{\sup_{s\in[0,1]} W_1^*(s) > u\right\}.$$

In view of [2][Rem 2.2] applied to the lower bound (3.2), we get as $u \to \infty$

$$\pi_\rho(c_1,c_2;u,au) \geq 2\mathbb{P}\{W_1^*(1) > u\}(1+o(1)) = \mathbb{P}\left\{\sup_{s\in[0,1]} W_1^*(s) > u\right\}(1+o(1))$$

and hence as $u \to \infty$

$$\pi_\rho(c_1,c_2;u,au) \sim \mathbb{P}\left\{\sup_{s\in[0,1]} W_1^*(s) > u\right\} = \pi_1^{(1)}(c_1;u).$$

Case $0 < a = \rho < 1$. The asymptotics of the lower bound follows again from [2][Rem 2.2] applied to (3.2), namely we have

$$\pi_\rho(c_1,c_2;u,\rho u) \geq \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right)\pi_1^{(1)}(c_1;u)(1+o(1)), \quad u\to\infty. \tag{3.3}$$

Setting $h_u := 1 - \frac{1}{\sqrt{u}}$ we have the following upper bound

$$\begin{aligned}\pi_\rho(c_1,c_2;u,\rho u) \quad &\leq \quad \mathbb{P}\big\{\exists_{s,t\in[h_u,1]} : W_1^*(s) > u, W_2^*(t) > \rho u\big\} + \mathbb{P}\big\{\exists_{s\in[0,h_u]} : W_1^*(s) > u\big\}\\ &\quad +\mathbb{P}\big\{\exists_{s\in[h_u,1],t\in[0,h_u]} : W_1^*(s) > u, W_2^*(t) > \rho u\big\}.\end{aligned}$$

It follows from (1.2) that for some $C \in (0,\infty)$

$$\mathbb{P}\big\{\exists_{s\in[0,h_u]} : W_1^*(s) > u\big\} \leq Ce^{-\frac{1}{2}u\sqrt{u}}\pi_1^{(1)}(c_1;u)(1+o(1)).$$

Additionally, for any $u > 0$ we have

$$\begin{aligned}&\mathbb{P}\big\{\exists_{s\in[h_u,1],t\in[0,h_u]} : W_1^*(s) > u, W_2^*(t) > \rho u\big\}\\ &\leq \mathbb{P}\left\{\exists_{s\in[h_u,1],t\in[0,h_u]} : \frac{b_{1,u}(s,t)B_1(s) + b_{2,u}(s,t)(\rho B_1(t) + \sqrt{1-\rho^2}B_2(t))}{b_{1,u}(s,t)(1+\frac{c_1 s}{u}) + b_{2,u}(s,t)}(\rho + \frac{c_2 t}{u}) > u\right\},\end{aligned}$$

where

$$\boldsymbol{b}_u(s,t) := (b_{1,u}(s,t), b_{2,u}(s,t)) = \Sigma_{s,t}^{-1}(1+\frac{c_1 s}{u}, \rho + \frac{c_2 t}{u})^\top.$$

Since for all u large we prove in the Appendix that

$$\sup_{s\in[h_u,1],t\in[0,h_u]} Var\left(\frac{b_{1,u}^*(s,t)B_1(s) + b_{2,u}^*(s,t)(\rho B_1(t) + \sqrt{1-\rho^2}B_2(t))}{b_{1,u}^*(s,t)(1+\frac{c_1 s}{u}) + b_{2,u}^*(s,t)(a + \frac{c_2 t}{u})}\right) = 1 - \frac{\rho^2}{1-\rho^2}\frac{1}{u}(1+o(1)), \tag{3.4}$$

then using [11][Thm 8.1] we obtain for some $C, \bar{C}, \epsilon$ positive and sufficiently large u

$$\begin{aligned}\mathbb{P}\{\exists_{s\in[h_u,1],t\in[0,h_u]} : W_1^*(s) > u, W_2^*(t) > \rho u\} &\le Cu^\epsilon e^{-\frac{u^2}{2}\frac{1}{1-\bar{C}\frac{1}{u}}}\\ &= o(\pi_1^{(1)}(c_1;u)),\ u\to\infty.\end{aligned}$$

Using the above, as $u \to \infty$

$$\begin{aligned}\pi_\rho(c_1,c_2;u,\rho u) &\le \mathbb{P}\{\exists_{s,t\in[h_u,1]} : W_1^*(s) > u, W_2^*(t) > \rho u\}(1+o(1))\\ &\le \left(\mathbb{P}\left\{\exists_{s,t\in[h_u,1]}:\ W_1^*(s) > u, W_2^*(t) > \rho u, \forall_{s\in[h_u,1]} : u + \frac{1}{\sqrt{u}} > W_1^*(s)\right\}\right.\\ &\quad \left. + \mathbb{P}\left\{\exists_{s\in[h_u,1]}:\ W_1^*(s) > u + \frac{1}{\sqrt{u}}\right\}\right)(1+o(1)).\end{aligned}$$

Thanks to (1.2) , for some $C > 0$ and sufficiently large u

$$\begin{aligned}\frac{\mathbb{P}\left\{\exists_{s\in[h_u,1]}:\ W_1^*(s) > u + \frac{1}{\sqrt{u}}\right\}}{\pi_1(c_1;u)} &\le \frac{\mathbb{P}\left\{\exists_{s\in[0,1]}:\ W_1^*(s) > u + \frac{1}{\sqrt{u}}\right\}}{\mathbb{P}\{W_1^*(1) > u\}}\\ &= \frac{\Phi\left(-(u+\frac{1}{\sqrt{u}}) - c_1\right) + e^{-2c_1(u+\frac{1}{\sqrt{u}})}\Phi\left(-(u+\frac{1}{\sqrt{u}}) + c_1\right)}{\Phi\left(-u-c_1\right)}\\ &\le Ce^{-\frac{(u+\frac{1}{\sqrt{u}})^2}{2}+\frac{u^2}{2}}\\ &= Ce^{-\sqrt{u}-\frac{1}{2u}}.\end{aligned}$$

Moreover, since $\rho > 0$, for $\bar{c_2} = c_2 - \rho c_1$

$$\begin{aligned}&\mathbb{P}\left\{\exists_{s,t\in[h_u,1]}:\ W_1^*(s) > u, W_2^*(t) > \rho u, \forall_{s\in[h_u,1]} : u + \frac{1}{\sqrt{u}} > W_1^*(s)\right\}\\ &\le \mathbb{P}\left\{\exists_{s,t\in[h_u,1]} : B_1(s) - c_1 s > u, \rho(u + \frac{1}{\sqrt{u}}) + \sqrt{1-\rho^2}B_2(t) - \bar{c_2}t > \rho u\right\}\\ &\le \mathbb{P}\{\exists_{s\in[0,1]} : B_1(s) - c_1 s > u\}\mathbb{P}\left\{\exists_{t\in[h_u,1]} : \sqrt{1-\rho^2}B_2(t) - \bar{c_2}t > -\rho\frac{1}{\sqrt{u}}\right\}\\ &= \mathbb{P}\{\exists s\in[0,1] : W_1^*(s) > u\}\Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right)(1+o(1)),\quad u\to\infty\end{aligned}$$

implying the following upper asymptotic bound

$$\pi_\rho(c_1,c_2;u,\rho u) \le \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right)\pi_1^{(1)}(c_1;u)(1+o(1)),$$

which combined with the asymptotic lower bound (3.3) completes the proof. ☐

3.3. **Proof of Theorem 2.2.** First we state several technical lemmas that are used in the proof. Suppose that $a \in (\max(0,\rho),1]$ and recall $\Sigma_{s,t}$ defined in (2.1). Denote below for $\boldsymbol{a} = (1,a)^\top$

$$q_{\boldsymbol{a}}(s,t) = \boldsymbol{a}^\top \Sigma_{s,t}^{-1} \boldsymbol{a} = \frac{t - 2a\rho\min(s,t) + a^2 s}{st - (\rho\min(s,t))^2},$$

$$\boldsymbol{b}(s,t) = \Sigma_{s,t}^{-1}\boldsymbol{a} = \frac{1}{st - (\rho\min(s,t))^2}(t - a\rho\min(s,t), as - \rho\min(s,t))^\top$$

and set

$$q^*_{\boldsymbol{a}}(s,t) = \min_{\boldsymbol{x}\geq \boldsymbol{a}} q_{\boldsymbol{x}}(s,t), \quad q^*_{\boldsymbol{a}} = \min_{s,t\in[0,1]} q^*_{\boldsymbol{a}}(s,t). \tag{3.5}$$

It is well-known that $q^*_{\boldsymbol{a}}(s,t)$ captures the asymptotics of $\mathbb{P}\{W_1^*(s) > u, W_2^*(t) > au\}$ in the sense that for any $s,t > 0$ we have the following logarythmic asymptotics

$$\lim_{u\to\infty} \frac{1}{u^2}\log \mathbb{P}\{W_1^*(s) > u, W_2^*(t) > au\} = -\frac{q^*_{\boldsymbol{a}}(s,t)}{2}. \tag{3.6}$$

Moreover, by [12], we have

$$\lim_{u\to\infty} \frac{1}{u^2}\log \mathbb{P}\{\exists_{s,t\in[0,1]} W_1^*(s) > u, W_2^*(t) > au\} = -\frac{q^*_{\boldsymbol{a}}}{2}. \tag{3.7}$$

However, in order to obtain the exact asymptotics, one has to take into account the drifts of W_1^*, W_2^*. Therefore a slightly redefined optimization problem has to be considered. Specifically, for $\boldsymbol{a}_u(s,t) = (1 + \frac{c_1 s}{u}, a + \frac{c_2 t}{u})^\top$ define

$$q_{\boldsymbol{a}_u(s,t)}(s,t) = \boldsymbol{a}_u(s,t)^\top \Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t), \quad q^*_{\boldsymbol{a}_u(s,t)}(s,t) = \min_{\boldsymbol{x}\geq \boldsymbol{a}_u(s,t)} q_{\boldsymbol{x}}(s,t)$$

and

$$\boldsymbol{b}_u(s,t) = (b_{1,u}(s,t), b_{2,u}(s,t)) =: \Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t).$$

Note that for $a \in (\max(0,\rho),1]$ and large enough u we have

$$\boldsymbol{b}(s,t) > (0,0)^\top, \quad \boldsymbol{b}_u(s,t) > (0,0)^\top.$$

Lemma 3.1. *For all large u we have:*

(i) If $a = 1, \rho \in (-1, -\frac{1}{2})$, then $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ attains its unique local minima on $[0,1]^2$ at

$$(s_u,t_u) := \left(1, \frac{1}{\rho(2\rho-1) + \frac{c_2 - \rho c_1}{u}}\right), \quad (\bar{s}_u,\bar{t}_u) := \left(\frac{1}{\rho(2\rho-1) + \frac{c_1 - \rho c_2}{u}}, 1\right).$$

(ii) If $a = 1, \rho = -\frac{1}{2}$, then $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ attains its unique local minima on $[0,1]^2$ at

$$(s_u,t_u) := \left(1, \min(\frac{1}{1 + \frac{c_2 + 2c_1}{u}}, 1)\right), \quad (\bar{s}_u,\bar{t}_u) := \left(\min(\frac{1}{1 + \frac{c_1 + 2c_2}{u}}, 1), 1\right).$$

(iii) For any other $a \in (\max(0,\rho), 1], \rho \in (-1,1)$, $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ *attains its unique minimum on* $[0,1]^2$ *at*

$$(s_u, t_u) := \begin{cases} (1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}), & if\ \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}} \in [0,1] \\ (1,1), & otherwise. \end{cases}$$

In the rest of the paper we set

$$t^* := \lim_{u\to\infty} t_u,$$

where t_u is defined as in Lemma 3.1.

Lemma 3.2. *For* $a \in (\max(0,\rho), 1]$ *and* t_u *as in Lemma 3.1 we have for large enough* u

$$\frac{u^2}{2}\left(q^*_{\boldsymbol{a}_u(1,t_u)}(1,t_u) - q_{\boldsymbol{a}_u(1,t^*)}(1,t^*)\right) = \kappa^* \frac{a(c_1^2\rho^2 - 2c_1c_2\rho + c_2^2)}{2\rho(1-a\rho)},$$

where $\kappa^* = 0$ *if* $t^* = t_u = 1$ *and* $\kappa^* = 1$ *otherwise.*

Remark 3.3. *Let* $t_u < 1$ *be as in Lemma 3.1. Then*

$$\varphi_{t_u}(u+c_1, au+c_2t_u) = \frac{1}{2\pi|\Sigma_{1,t_u}|}e^{-\frac{1}{2}(u+c_1,au+c_2t_u)\Sigma^{-1}_{1,t_u}(u+c_1,au+c_2t_u)^\top} = \frac{1}{2\pi|\Sigma_{1,t_u}|}e^{-u^2\frac{q^*_{\boldsymbol{a}_u(1,t_u)}(1,t_u)}{2}},$$

$$\varphi_{t^*}(u+c_1, au+c_2t^*) = \frac{1}{2\pi|\Sigma_{1,t^*}|}e^{-\frac{1}{2}(u+c_1,au+c_2t^*)\Sigma^{-1}_{1,t^*}(u+c_1,au+c_2t^*)^\top} = \frac{1}{2\pi|\Sigma_{1,t^*}|}e^{-u^2\frac{q_{\boldsymbol{a}_u(1,t^*)}(1,t^*)}{2}}.$$

Hence it follows from Lemma 3.2 that for $a \in (\max(0,\rho), 1]$

$$\varphi_{t_u}(u+c_1, au+c_2t_u) \sim e^{-a\frac{(c_1\rho-c_2)^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1, au+c_2t^*),\ u\to\infty.$$

Hereafter we use the notation for optimizers of $\min_{s,t\in[0,1]} q^*_{\boldsymbol{a}_u(1,t_u)}(s,t)$ as introduced in Lemma 3.1. Due to the symmetry of the case (vi) in Theorem 2.2 (note that $a=1$ in this case), in the rest of lemmas presented below we focus only on the analysis of local properties of (W_1^*, W_2^*) in the neighbourhood of (s_u, t_u), where $s_u = 1$.

Let in the following

$$k_u = 1 - \frac{(k-1)\Delta}{u^2}, l_u = t_u - \frac{(l-1)\Delta}{u^2},\ u>0, \Delta>0$$

and set

$$E_{u,k} = [(k+1)_u, k_u], \quad E_{u,k,l} = E_{u,k}\times E_{u,l}, \quad E = [-\Delta,0]\times[-\Delta,0].$$

Define also

$$\chi_{u,k,l}(s,t) := (\chi_{1,u,k}(s), \chi_{2,u,l}(t)) := u\left(W_1(\frac{s}{u^2}+k_u) - W_1(k_u) - c_1\frac{s}{u^2}, W_2(\frac{t}{u^2}+l_u) - W_2(l_u) - c_2\frac{t}{u^2}\right).$$

Lemma 3.4. *If $\rho \in (-1,1), a \in (\max(0,\rho),1], l,k \le \frac{u\log(u)}{\Delta}$ and $\Delta > 0$ are given constants, then*

$$\mathbb{P}\left\{ \exists_{(s,t)\in E_{u,k,l}} : \begin{array}{c} W_1^*(s) > u \\ W_2^*(t) > au \end{array} \right\} \sim I(\Delta) u^{-2} \varphi_{t_u}(u+c_1, au+c_2t_u) e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u) - q_{\boldsymbol{a}_u(1,t_u)}(1,t_u))}$$

as $u\to\infty$, where

$$I(\Delta) = \begin{cases} \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{s,t\in[0,\Delta]} : W_1(s) - s > x, W_2(t) - at > y\right\} e^{\lambda_1 x + \lambda_2 y} dxdy, & l_u = k_u \\ \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - s > x\right\} e^{\lambda_1 x} dx \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in[0,\Delta]} : W_2(t) - \frac{a-\rho}{t^*-\rho^2} t > y\right\} e^{\lambda_2 y} dy, & l_u > k_u \\ \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - \frac{1-a\rho}{1-\rho^2 t^*} s > x\right\} e^{\lambda_1 x} dx \int_{\mathbb{R}} \mathbb{P}\left\{\exists_{t\in[0,\Delta]} : W_2(t) - \frac{a}{t^*} t > y\right\} e^{\lambda_2 y} dy, & l_u < k_u \end{cases}$$

and $\lambda_1 = \begin{cases} \frac{1}{t^*}\frac{1-a\rho}{1-\rho^2} & l_u = k_u \\ \frac{t^*-a\rho}{t^*-\rho^2}, & l_u > k_u \\ \frac{1-a\rho}{1-\rho^2 t^*}, & l_u < k_u \end{cases}$, $\lambda_2 = \begin{cases} \frac{1}{t^*}\frac{a-\rho}{1-\rho^2}, & l_u = k_u \\ \frac{a-\rho}{t^*-\rho^2}, & l_u > k_u \\ \frac{a-\rho t^*}{t^*-\rho^2(t^*)^2}, & l_u < k_u \end{cases}$.

Additionally

$$\lim_{u\to\infty} \sup_{l,k=O(u\log u)} \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{(s,t)\in E} : \ \chi_{u,k,l}(s,t) > (x,y) \ \middle| \ \begin{array}{c} W_1^*(k_u) = u - \frac{x}{u} \\ W_2^*(l_u) = au - \frac{y}{u} \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy < \infty. \tag{3.8}$$

The proof of Lemma 3.4 is derived mainly by utilising the same idea as in the proof of the classical Pickands lemma, see e.g., Lemma D.1 in [11] or more recent contributions [8, 13]. We note that finiteness of (3.8) is important in the proof of Theorem 2.2, where in order to evaluate sum over many small intervals we apply similar technique to the one used in [14][Lem 2].

Lemmas 3.5, 3.6 deal with $\lim_{\Delta\to\infty} I(\Delta)$ for $l_u \neq k_u$ and $l_u = k_u$ (see Lemma 3.4) respectively. The proof of Lemma 3.5 follows straightforwardly from (1.2), while the proof of Lemma 3.6 is largely the same as the proof of finiteness of two-dimensional Piterbarg-type constants given in [2], see also [8].

Lemma 3.5. *Let in the following B be a standard Brownian motion.*

i) For all $b,c>0$ such that $2b>c$ we have

$$\lim_{\Delta\to\infty} \int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,\Delta]} (B(t) - bt) > x\} e^{cx} dx = \frac{1}{2b-c} + \frac{1}{c}.$$

ii) For all $b>0$

$$\lim_{\Delta\to\infty} \frac{1}{\Delta} \int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,\Delta]} (B(t) - bt) > x\} e^{2bx} dx = b.$$

Define for any $a > \max(0,\rho), \boldsymbol{q} \in \mathbb{R}^2$, $\Delta \in (0,\infty)$ and $\Sigma = \Sigma_{1,1}, \boldsymbol{a} = (1,a)^\top$

$$I(\Delta, \boldsymbol{q}) := \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{\boldsymbol{s}\in[0,\Delta]^2} : \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{q}\cdot\boldsymbol{s} > \boldsymbol{x}\right\} e^{\boldsymbol{a}^\top \Sigma^{-1}\boldsymbol{x}} d\boldsymbol{x},$$

where $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes component-wise multiplication of vectors. Note that $I(\Delta,\boldsymbol{a})$ is the constant $I(\Delta)$ that appears in the case $k_u = l_u$ in Lemma 3.4.

Lemma 3.6. *For any $a \in (\max(0,\rho),1]$ we have $I(\Delta,\boldsymbol{q}) \in (0,\infty)$ and*

$$\lim_{\Delta\to\infty} I(\Delta,\boldsymbol{a}) \in (0,\infty). \tag{3.9}$$

Next for $C>0, i,j\in\mathbb{R}$ let

$$Q_{i,j} = \sum_{l=[i]}^{[j]} \frac{\sqrt{C}\Delta}{u} e^{-\frac{C_1}{2}\frac{(l-1)^2\Delta^2}{u^2}},$$

where $[n]$ denotes the integer part of n. The following asymptotic result is used several times in the proof of the main theorem.

Lemma 3.7. *For $c>0$ and $C>0$ from the definition of $Q_{i,j}$ above we have*

$$\lim_{\Delta\to\infty}\lim_{u\to\infty} Q_{1,\frac{u\log u}{\Delta}} = \frac{\sqrt{\pi}}{\sqrt{2}}, \quad \lim_{\Delta\to\infty}\lim_{u\to\infty} Q_{-\frac{u\log u}{\Delta},\frac{u\log u}{\Delta}} = \sqrt{2\pi} \quad \lim_{\Delta\to\infty}\lim_{u\to\infty} Q_{-\frac{cu}{\Delta},\frac{u\log u}{\Delta}} = \sqrt{2\pi}\Phi\left(c\right).$$

Since the proofs of the above two lemmas follow by straightforward calculations, they are omitted.

Proof of Theorem 2.2 In the first part of the proof we show that in order to determine the exact asymptotics of π_ρ one can restrict the parameter set $[0,1]^2$ to the area of size $\frac{\log u}{u}$ around the optimising points that were found in Lemma 3.1. The proof is split into six cases; in each case the contributing interval responsible for the asymptotics is different and a bit different argument has to be used. In cases (i)-(iii) and (v) there is one point minimizing the function $q_{\boldsymbol{a}(s,t)}(s,t)$ and the analysis focuses on the neighbourhood of this point. In cases (iv) and (vi) there are two points minimizing the function $q_{\boldsymbol{a}(s,t)}(s,t)$ and hence we treat this cases differently.

Define

$$F_u = [1-\frac{\log u}{u},1]\times[t_u-\frac{\log u}{u},\min(t_u+\frac{\log u}{u},1)].$$

Let next

$$N_u := \lfloor \frac{u\log(u)}{\Delta}\rfloor, \quad K_u^{(1)} = \frac{(c_2-c_1\rho)u}{\Delta}, \quad K_u^{(2)} = \frac{(c_1-c_2\rho)u}{\Delta},$$
$$E^1_{u,m} := [(m+1)_u, m_u], \quad E^2_{u,j} := [(j+1)_u, j_u],$$

where

$$m_u = 1-(m-1)\Delta u^{-2}, \quad j_u = t_u-(j-1)\Delta u^{-2}, \ \Delta>0.$$

In the first step, which is common for cases (i)-(iii) and (v), we observe that

$$\pi_\rho(c_1,c_2;u,au) = \mathbb{P}\{\exists (s,t)\in F_u:\ W_1^*(s)>u, W_2^*(t)>au\}(1+o(1)) \tag{3.10}$$

$$\leq \sum_{k=1}^{N_u}\sum_{l=1-\mathbf{1}(t_u<1)(N_u+1)}^{N_u} \mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}}: W_1^*(s)>u, W_2^*(t)>au\Big\}(1+o(1)) \tag{3.11}$$

as $u \to \infty$, where (3.10) follows from [11][Thm 8.1] and is proven in detail in Appendix, while (3.11) is due to Bonferroni inequality.

Case (i): $\rho > \frac{1}{4a}(1-\sqrt{8a^2+1})$. According to Lemma 3.1 there is exactly one minimizer of $q^*_{\boldsymbol{a}(s,t)}(s,t)$ on $[0,1]^2$: $(s_u,t_u)=(1,1)$. Our aim is to prove that for $F_u = E^2_{u,1}$

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\pi_\rho(c_1,c_2;u,au)}{\mathbb{P}\{\exists_{s,t\in F_u}: W_1^*(s)>u, W_2^*(t)>au\}} = 1. \tag{3.12}$$

Lower bound. For $\Delta>0$, by Lemma 3.4, we have

$$\begin{aligned}\pi_\rho(c_1,c_2;u,au) &\geq \mathbb{P}\Big\{\exists_{s,t\in[1-\frac{\Delta}{u^2},1]}: W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &\sim I(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2),\ u\to\infty,\end{aligned}$$

where

$$I(\Delta)=\int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{s,t\in[0,\Delta]}: \begin{array}{c} W_1(s)-s>x\\ W_2(t)-at>y\end{array}\right\}e^{\lambda_1 x+\lambda_2 y}dxdy<\infty,\qquad \lambda_1=\frac{1-a\rho}{1-\rho^2}>0, \lambda_2=\frac{a-\rho}{1-\rho^2}>0.$$

Upper bound. By (3.11), we get as $u\to\infty$

$$\begin{aligned}\pi_\rho(c_1,c_2;u,au) &\leq \Big(\mathbb{P}\Big\{\exists_{s\in E^1_{u,1},t\in E^2_{u,1}}: W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &+ \sum_{k=2}^{N_u}\sum_{l=1}^{k-1}\mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}}: W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &+ \sum_{l=2}^{N_u}\sum_{k=1}^{l-1}\mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}}: W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &+ \sum_{k=2}^{N_u}\mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,k}}: W_1^*(s)>u, W_2^*(t)>au\Big\}\Big)(1+o(1))\\ &:= S_1+S_2+S_3+S_4.\end{aligned}$$

Using Lemma 3.4 we get as $u\to\infty$

$$S_1 \sim I(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2).$$

Before proceeding to the analysis of S_2,S_3,S_4 observe that as $u\to\infty$

$$u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,1)}(1,1)) = \tau_1(k-1)\Delta+\tau_2(l-1)\Delta+O\left(\frac{\Delta^2(k+l)^2}{u^4}\right), \tag{3.13}$$

where

$$\tau_1=\begin{cases}\tau_1^{(1)}:=\frac{(1-a\rho)^2}{(1-\rho^2)^2} & k<l\\ \tau_1^{(2)}:=\frac{1-2\rho^2+2a\rho^3-a^2\rho^2}{(1-\rho^2)^2} & k>l\end{cases},\ \tau_2=\begin{cases}\tau_2^{(1)}:=\frac{-\rho^2+2a\rho^3+a^2-2a^2\rho^2}{(1-\rho^2)^2} & k<l\\ \tau_2^{(2)}:=\frac{(a-\rho)^2}{(1-\rho^2)^2} & k>l\end{cases},$$

$\tau_1, \tau_2 > 0$.

Observe that since $k, l = O(\frac{u \log u}{\Delta})$, hence with Lemma 3.4 we obtain as $u \to \infty$

$$\begin{aligned} S_2 &= \left(\sum_{k=2}^{N_u} \sum_{l=1}^{k-1} e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}}(k_u,l_u)-q_{\boldsymbol{a}}(1,1))} \right) I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)) \\ &= (1+o(1)) \sum_{k=2}^{N_u} \sum_{l=1}^{k-1} e^{-\frac{1}{2}\tau_1^{(2)}(k-1)\Delta} e^{-\frac{1}{2}\tau_2^{(2)}(l-1)\Delta} I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2) \\ &\leq \frac{e^{-\frac{1}{2}\tau_1^{(2)}\Delta}}{1-e^{-\frac{1}{2}\tau_1^{(2)}\Delta}} \frac{e^{-\frac{1}{2}\tau_2^{(2)}\Delta}}{1-e^{-\frac{1}{2}\tau_2^{(2)}\Delta}} I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)). \end{aligned}$$

Similarly, by Lemma 3.4, we get as $u \to \infty$

$$\begin{aligned} S_3 &\leq (1+o(1)) \sum_{l=2}^{N_u} \sum_{k=1}^{l-1} e^{-\frac{1}{2}\tau_1^{(1)}(k-1)\Delta} e^{-\frac{1}{2}\tau_2^{(1)}(l-1)\Delta} I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2) \\ &\leq \frac{e^{-\frac{1}{2}\tau_1^{(1)}\Delta}}{1-e^{-\frac{1}{2}\tau_1^{(1)}\Delta}} \frac{e^{-\frac{1}{2}\tau_2^{(1)}\Delta}}{1-e^{-\frac{1}{2}\tau_2^{(1)}\Delta}} I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)). \end{aligned}$$

Finally we have that as $u \to \infty$

$$\begin{aligned} S_4 &\leq \sum_{k=2}^{N_u} \mathbb{P}\left\{ \exists_{s \in E^1_{u,k}, t \in E^2_{u,k}}, s < t : W_1^*(s) > u, W_2^*(t) > au \right\} \\ &\quad + \sum_{k=2}^{N_u} \mathbb{P}\left\{ \exists_{s \in E^1_{u,k}, t \in E^2_{u,k}}, s > t : W_1^*(s) > u, W_2^*(t) > au \right\} \\ &\quad + \mathbb{P}\left\{ \exists_{s \in [1-\frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}]} : W_1^*(s) > u, W_2^*(s) > au \right\} \\ &\leq \left(\sum_{k=2}^{N_u} e^{-\tau_1^{(1)}(k-1)\Delta} e^{-\tau_2^{(1)}(k-1)\Delta} + e^{-\tau_1^{(2)}(k-1)\Delta} e^{-\tau_2^{(2)}(k-1)\Delta} \right) I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)) \\ &\quad + \mathbb{P}\left\{ \exists_{s \in [1-\frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}]} : W_1^*(s) > u, W_2^*(s) > au \right\} \\ &\leq \left(\frac{e^{-(\tau_1^{(1)}+\tau_2^{(1)})\Delta}}{1-e^{-(\tau_1^{(1)}+\tau_2^{(1)})\Delta}} + \frac{e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}}{1-e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}} \right) I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)) \\ &\quad + O(e^{-\frac{\Delta}{8}} u^{-2} \varphi_1(u+c_1, au+c_2)) \qquad (3.14) \\ &= \left(\frac{e^{-(\tau_1^{(1)}+\tau_2^{(1)})\Delta}}{1-e^{-(\tau_1^{(1)}+\tau_2^{(1)})\Delta}} + \frac{e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}}{1-e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}} + e^{-\frac{\Delta}{8}} \right) I(\Delta) u^{-2} \varphi_1(u+c_1, au+c_2)(1+o(1)), \end{aligned}$$

where in (3.14) we use [2][Lem 3.1]. Further by Lemma 3.6 we have

$$C_1 := \lim_{\Delta \to \infty} I(\Delta) = \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{s,t \in [0,\infty)} : \begin{array}{c} W_1(s) - s > x \\ W_2(t) - at > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dx dy \in (0, \infty).$$

Hence we get

$$\lim_{\Delta \to \infty} \lim_{u \to \infty} \frac{S_2 + S_3 + S_4}{S_1} = 0.$$

Therefore

$$\lim_{u\to\infty} \frac{\pi_\rho(c_1,c_2;u,au)}{C_1 u^{-2}\varphi_1(u+c_1,au+c_2)} = \lim_{u\to\infty} \frac{S_1}{C_1 u^{-2}\varphi_1(u+c_1,au+c_2)} = 1.$$

Case (ii): $\rho = \frac{1}{4a}(1-\sqrt{8a^2+1}), c_2-\rho c_1 \le 0$. According to Lemma 3.1 there is exactly one minimizer of $q^*_{\boldsymbol{a}(s,t)}(s,t)$ on $[0,1]^2$: $(s_u,t_u)=(1,1)$.

Lower bound. By Bonferroni inequality we have that for $u\to\infty$

$$\begin{aligned}
&\pi_\rho(c_1,c_2;u,au)\\
&\ge \mathbb{P}\left\{\exists (s,t)\in[1-\frac{\Delta}{u^2},1]\times[1-\frac{\log(u)}{u},1]:\ W_1^*(s)>u, W_2^*(t)>au\right\}\\
&\ge \sum_{l=2}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t\in E^2_{u,l}} : W_1^*(s)>u, W_2^*(t)>au\right\}\\
&\quad -\sum_{l=2}^{N_u}\sum_{m=l+1}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,l},t_2\in E^2_{u,m}} : W_1^*(s)>u, W_2^*(t_1)>au, W_2^*(t_2)>au\right\}. \qquad (3.15)
\end{aligned}$$

Using direct calculations for $k<l$ we have that

$$u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,1)}(1,1)) = \tau_1^{(1)}(k-1)\Delta+\tau_4\frac{(l-1)^2\Delta^2}{u^2}+o\left(\frac{k^2}{u^2}\right)+o\left(\frac{l^3}{u^4}\right),$$

where $\tau_1^{(1)} = \frac{(1-a\rho)^2}{(1-\rho^2)^2}>0$ and $\tau_4 = \frac{\rho^2-2a\rho^3+a^2\rho^2}{(1-\rho^2)^2}>0$ and for $k>l$ we have that

$$u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,1)}(1,1)) = \tau_1^{(2)}(k-1)\Delta+\tau_2(l-1)\Delta+o\left(\frac{k^2}{u^2}\right)+o\left(\frac{l^2}{u^2}\right),$$

where $\tau_1^{(2)} = \frac{1-a^2\rho^2+2a\rho^3-2\rho^2}{(1-\rho^2)^2}>0$ and $\tau_2=\frac{(a-\rho)^2}{(1-\rho^2)^2}>0$.

By Lemma 3.4, as $u\to\infty$, we have

$$\begin{aligned}
&\sum_{l=2}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t\in E^2_{u,l}} : W_1^*(s)>u, W_2^*(t)>au\right\}\\
&\sim I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u} e^{-\frac12 u^2(q_{\boldsymbol{a}}(k_u,l_u)-q_{\boldsymbol{a}}(1,1))}\\
&\sim I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&= I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&\le I_1(\Delta)\frac{I_2(\Delta)}{\Delta}u^{-1}\frac{1}{\sqrt{\tau_4}}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}},
\end{aligned}$$

where

$$I_1(\Delta)=\int_{\mathbb{R}}\mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s)-\frac{1-a\rho}{1-\rho^2}s>x\right\}e^{\frac{1-a\rho}{1-\rho^2}x}dx, \quad I_2(\Delta)=\int_{\mathbb{R}}\mathbb{P}\{\sup_{t\in[0,\Delta]}(W_2(t)-at)>x\}e^{2ax}dx.$$

In view of Lemma 3.5 $\lim_{\Delta\to\infty} I_1(\Delta) = 2\frac{1-\rho^2}{1-a\rho}$ and $\lim_{\Delta\to\infty}\frac{I_2(\Delta)}{\Delta} = a$. Since with Lemma 3.7 we have that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\sum_{l=2}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}} = \frac{\sqrt{\pi}}{\sqrt{2}}2\tau_4,$$

hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\sum_{l=2}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}}{a\frac{1-\rho^2}{1-a\rho}\frac{\sqrt{2\pi}}{\sqrt{\tau_4}}u^{-1}\varphi_1(u+c_1, au+c_2)} \quad = \quad 1.$$

In order to complete the proof of the lower bound, (3.15) needs to be shown to be asymptotically negligible as $\Delta\to\infty$ and then $u\to\infty$, which follows by standard calculations as in e.g., [11] [Section D]. We defer precise calculations to Appendix.

Upper bound. Using (3.11) we have for any $\Delta > 0$, as $u\to\infty$

$$\begin{aligned}
\pi_\rho(c_1,c_2;u,au) \quad &\le \quad \Big(\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t\in E^2_{u,1}} : W_1^*(s) > u, W_2^*(t) > au\right\}\\
&+ \quad \sum_{k=2}^{N_u}\sum_{l=1}^{k-1}\mathbb{P}\left\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}\\
&+ \quad \sum_{l=2}^{N_u}\sum_{k=1}^{l-1}\mathbb{P}\left\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}\\
&+ \quad \sum_{k=2}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,k},t\in E^2_{u,k}} : W_1^*(s) > u, W_2^*(t) > au\right\}\Big)(1+o(1))\\
&:= \quad S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

Further, by Lemma 3.4

$$\begin{aligned}
S_3 \quad &\sim \quad I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u}\sum_{k=1}^{l-1}e^{-\frac{\tau_1^{(1)}}{2}(k-1)\Delta}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&= \quad I_1(\Delta)I_2(\Delta)\frac{1}{1-e^{-\frac{\Delta\tau_1^{(1)}}{2}}}u^{-2}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u}(1-e^{-\frac{\Delta\tau_1^{(1)}(l-1)}{2}})e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&\le \quad I_1(\Delta)\frac{I_2(\Delta)}{\Delta}\frac{1}{1-e^{-\frac{\Delta\tau_1^{(1)}}{2}}}u^{-1}\frac{1}{\sqrt{\tau_4}}\varphi_1(u+c_1,au+c_2)\sum_{l=2}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}.
\end{aligned}$$

Hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\sum_{l=2}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}}{S_3} = 1.$$

Lemma 3.4 implies that as $u\to\infty$

$$S_2 \quad \le \quad \sum_{k=2}^{N_u}\sum_{l=1}^{k-1}e^{-\tau_1^{(2)}(k-1)\Delta}e^{-\tau_2(l-1)\Delta}I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)(1+o(1))$$

$$\leq \frac{e^{-\tau_1^{(2)}\Delta}}{1-e^{-\tau_1^{(2)}\Delta}} \frac{e^{-\tau_2\Delta}}{1-e^{-\tau_2\Delta}} I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)(1+o(1)).$$

Again by Lemma 3.4 for some $C>0$ as $u\to\infty$

$$\begin{aligned} S_4 &\leq \sum_{k=2}^{N_u} \mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,k}}, s<t : W_1^*(s)>u, W_2^*(t)>au\Big\} \\ &\quad + \sum_{k=2}^{N_u} \mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,k}}, s>t : W_1^*(s)>u, W_2^*(t)>au\Big\} \\ &\quad + \mathbb{P}\Big\{\exists_{s\in[1-\frac{\log u}{u},1-\frac{\Delta}{u^2}]} : W_1^*(s)>u, W_2^*(s)>au\Big\} \\ &\leq \sum_{k=2}^{N_u} (e^{-\frac{\tau_1^{(1)}}{2}(k-1)\Delta} e^{-\frac{\tau_4}{2}\frac{(k-1)^2\Delta^2}{u^2}} + e^{-\tau_1^{(2)}(k-1)\Delta}e^{-\tau_2(k-1)\Delta}) \\ &\quad \times I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)(1+o(1)) \\ &\quad + \mathbb{P}\Big\{\exists_{s\in[1-\frac{\log u}{u},1]} : W_1^*(s)>u, W_2^*(s)>au\Big\}. \end{aligned}$$

Further, using [2][Thm 2.1] we obtain

$$\begin{aligned} S_4 &\leq \left(e^{-\frac{\tau_1^{(1)}}{2}\Delta}\sum_{k=2}^{N_u} e^{-\frac{\tau_4}{2}\frac{(k-1)^2\Delta^2}{u^2}} + \frac{e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}}{1-e^{-\tau_1^{(2)}\Delta}}\right) \\ &\quad \times I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)(1+o(1)) + Cu^{-2}\varphi_1(u+c_1,au+c_2) \\ &\leq \left(e^{-\frac{\tau_1^{(1)}}{2}\Delta}S_3 + \frac{e^{-(\tau_1^{(2)}+\tau_2^{(2)})\Delta}}{1-e^{-\tau_1^{(2)}\Delta}} I_1(\Delta)I_2(\Delta)u^{-2}\varphi_1(u+c_1,au+c_2)\right)(1+o(1)). \end{aligned}$$

Finally notice that Lemma 3.4 implies that $S_1 = O(u^{-2}\varphi_1(u+c_1,au+c_2))$. Therefore

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{S_1+S_2+S_4}{S_3} = 0.$$

Note that from the proof we can see that (3.12) holds for $F_u = [1-\frac{\Delta}{u^2},1]\times[1-\frac{\log(u)}{u},1-\frac{\Delta}{u^2}]$.

Case (iii): $\rho = \frac{1}{4a}(1-\sqrt{8a^2+1}), c_2-\rho c_1>0$. According to Lemma 3.1 there is exactly one minimizer of $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ on $[0,1]^2$: $(s_u,t_u) = \left(1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\right)$, where $c_1-\rho c_2>0$ implies the important fact that

$$\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}} \nearrow 1, \quad u\to\infty.$$

Upper bound. Using (3.11) for all $\Delta>0$ as $u\to\infty$ we have

$$\begin{aligned} \pi_\rho(c_1,c_2;u,au) &\leq \Big(\mathbb{P}\Big\{\exists_{s\in E^1_{u,1},t\in E^2_{u,1}} : W_1^*(s)>u, W_2^*(t)>au\Big\} \\ &\quad + \sum_{k=2}^{N_u}\sum_{l=-K_u^{(1)}}^{k-K_u^{(1)}-1} \mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}} : W_1^*(s)>u, W_2^*(t)>au\Big\} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=-K_u^{(1)}}^{N_u} \sum_{k=1}^{l+K_u^{(1)}} \mathbb{P}\Big\{\exists_{s\in E_{u,k}^1, t\in E_{u,l}^2} : W_1^*(s) > u, W_2^*(t) > au\Big\} \\
&+ \sum_{k=-K_u^{(1)}}^{N_u-K_u^{(1)}} \mathbb{P}\Big\{\exists_{s\in E_{u,k}^1, t\in E_{u,k}^2} : W_1^*(s) > u, W_2^*(t) > au\Big\}\Big)(1+o(1)) \\
&:= S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

Further by Lemma 3.4 we have

$$
\begin{aligned}
S_3 &\sim I_1(\Delta) I_2(\Delta) u^{-2} e^{-a\frac{c_1^2\rho^2 - 2c_1c_2\rho + c_2^2}{2\rho(1-a\rho)}} \varphi_1(u+c_1, au+c_2) \sum_{l=-K_u^{(1)}}^{N_u} \sum_{k=1}^{l+K_u^{(1)}} e^{-\frac{\tau_1^{(1)}}{2}(k-1)\Delta} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}} \\
&= I_1(\Delta) I_2(\Delta) \frac{1}{1-e^{-\frac{\Delta\tau_1^{(1)}}{2}}} u^{-2} e^{-a\frac{c_1^2\rho^2 - 2c_1c_2\rho + c_2^2}{2\rho(1-a\rho)}} \varphi_1(u+c_1, au+c_2) \\
&\quad \times \sum_{l=-K_u^{(1)}}^{N_u} (1 - e^{-\frac{\Delta\tau_1^{(1)}(l+K_u^{(1)}-1)}{2}}) e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}} \\
&\le I_1(\Delta) \frac{I_2(\Delta)}{\Delta} \frac{1}{1-e^{-\frac{\Delta\tau_1^{(1)}}{2}}} u^{-1} \frac{1}{\sqrt{\tau_4}} e^{-a\frac{c_1^2\rho^2 - 2c_1c_2\rho + c_2^2}{2\rho(1-a\rho)}} \varphi_1(u+c_1, au+c_2) \\
&\quad \times \sum_{l=-K_u^{(1)}}^{N_u} \frac{\sqrt{\tau_4}\Delta}{u} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}},
\end{aligned}
$$

with $I_1(\Delta), I_2(\Delta)$ as in case (ii). Using Lemma 3.7 we have that

$$
\lim_{\Delta\to\infty} \lim_{u\to\infty} \frac{S_3}{\frac{2a}{\lambda_1} e^{-a\frac{c_1^2\rho^2 - 2c_1c_2\rho + c_2^2}{2\rho(1-a\rho)}} \frac{\sqrt{2\pi}}{\sqrt{\tau_4}} \Phi\left(c_2 - \rho c_1\right) u^{-1} \varphi_1(u+c_1, au+c_2)} = 1.
$$

Observe that S_1, S_2, S_4 up to a constant have been already calculated in case (ii) and hence

$$
\lim_{\Delta\to\infty} \lim_{u\to\infty} \frac{S_1+S_2+S_4}{S_3} = 0.
$$

Lower bound. For all $u > 0$

$$
\begin{aligned}
\pi_\rho(c_1, c_2; u, au) &\ge \mathbb{P}\Big\{\exists (s,t) \in [1-\frac{\Delta}{u^2}, 1] \times [t_u - \frac{\log(u)}{u}, 1] : \ W_1^*(s) > u, W_2^*(t) > au\Big\} \\
&\ge \sum_{l=-K_u^{(1)}}^{N_u} \mathbb{P}\Big\{\exists_{s\in E_{u,1}^1, t\in E_{u,l}^2} : W_1^*(s) > u, W_2^*(t) > au\Big\} \\
&\quad - \sum_{l=-K_u^{(1)}}^{N_u} \sum_{m=l+1}^{N_u} \mathbb{P}\Bigg\{\exists_{s\in E_{u,1}^1, t_1\in E_{u,l}^2, t_2\in E_{u,m}^2} : W_1^*(s) > u, W_2^*(t_1) > au, W_2^*(t_2) > au\Bigg\}. \qquad (3.16)
\end{aligned}
$$

We have that (3.16) is asymptotically negligible (see proof of negligibility of (3.15)). Hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\pi_\rho(c_1,c_2;u,au)}{\frac{2a}{\lambda_1}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\frac{\sqrt{2\pi}}{\sqrt{\tau_4}}\Phi\left(c_2-\rho c_1\right)u^{-1}\varphi_1(u+c_1,au+c_2)}=1,$$

which completes the proof of case (iii). Note that from the proof we can see that (3.12) holds for $F_u = [1-\frac{\Delta}{u^2},1]\times[t_u-\frac{\log(u)}{u},1-\frac{\Delta}{u^2}]$.

Case (iv): $a=1,\rho=-\frac{1}{2}$. According to Lemma 3.1 the minimizers of $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ on $[0,1]^2$ are

$$(s_u,t_u):=\left(1,\min(\frac{1}{1+\frac{c_2+2c_1}{u}},1)\right),\quad (\bar{s}_u,\bar{t}_u):=\left(\min(\frac{1}{1+\frac{c_1+2c_2}{u}},1),1\right)$$

and they both tend to $(s^*,t^*)=(1,1)$ as $u\to\infty$.

Upper bound. With (3.11) we can write

$$\begin{aligned}\pi_\rho(c_1,c_2;u,u) &\le \mathbb{P}\left\{\exists_{t\in[1-\frac{\log u}{u},1],s\in(t,1]}: W_1^*(s)>u, W_2^*(t)>u\right\}\\ &\quad+\mathbb{P}\left\{\exists_{s\in[1-\frac{\log u}{u},1],t\in(s,1]}: W_1^*(s)>u, W_2^*(t)>u\right\}\\ &\quad+\mathbb{P}\left\{\exists_{s\in[1-\frac{\log u}{u},1]}: W_1^*(s)>u, W_2^*(s)>u\right\}:=S_1+S_2+S_3.\end{aligned}$$

With the same calculations as in case (ii) and case (iii) we have that as $u\to\infty$

$$\begin{aligned}S_i &\sim \sum_{l=L_{1,u}}^{N_u-L_{1,u}}\mathbb{P}\left\{\exists_{s\in[1-\frac{\Delta}{u^2},1],t\in[t_u-\frac{(l-1)\Delta}{u^2},t_u-\frac{l\Delta}{u^2}]}: W_1^*(s)>u, W_2^*(t)>u\right\}\\ &\sim I_1(\Delta)\frac{I_2(\Delta)}{\Delta}\frac{1}{1-e^{-\frac{\Delta\tau_1^{(1)}}{2}}}u^{-1}\frac{1}{\sqrt{\tau_4}}e^{-\mathbf{1}(t_u<1)a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_1(u+c_1,au+c_2)\\ &\quad\times\sum_{l=L_{1,u}}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}},\end{aligned}$$

where $L_{1,u}=\begin{cases}3 & ,c_2+2c_1\le 0\\ -K_u^{(1)}+2 & ,c_2+2c_1>0\end{cases}$ and analogously we define $L_{2,u}=\begin{cases}3 & ,c_1+2c_2\le 0\\ -K_u^{(2)}+2 & ,c_1+2c_2>0\end{cases}$, which leads to

$$S_1\sim\begin{cases}\frac{\sqrt{2\pi}}{\sqrt{\tau}}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\Phi\left(c_2-\rho c_1\right)u^{-1}\varphi_1(u+c_1,u+c_2), & L_{1,u}=-K_u^{(1)}\\ \frac{\sqrt{\pi}}{\sqrt{2\tau}}u^{-1}\varphi_1(u+c_1,u+c_2), & L_{1,u}=3\end{cases}$$

as $\Delta\to\infty,u\to\infty$, where from case (ii) $\tau=\tau_4=\frac{4}{3}$. Analogously we get for $\Delta\to\infty,u\to\infty$

$$S_2\sim\begin{cases}\frac{\sqrt{2\pi}}{\sqrt{\tau}}e^{-a\frac{c_1^2-2c_1c_2\rho+c_2^2\rho^2}{2\rho(1-a\rho)}}\Phi\left(c_1-\rho c_2\right)u^{-1}\varphi_1(u+c_1,u+c_2), & L_{2,u}=-K_u^{(2)}\\ \frac{\sqrt{\pi}}{\sqrt{2\tau}}u^{-1}\varphi_1(u+c_1,u+c_2), & L_{2,u}=3\end{cases}.$$

The exact values of $L_{1,u}, L_{2,u}$ depend on the position of s_u, t_u with respect to $(1,1)$. On the other hand

$$\begin{aligned}\pi_\rho(c_1,c_2;u,u) \geq & \sum_{k=L_{1,u}}^{N_u-L_{1,u}} \mathbb{P}\Big\{\exists_{s\in[\bar{s}_u-\frac{(k-1)\Delta}{u^2},\bar{s}_u-\frac{k\Delta}{u^2}],t\in[1-\frac{\Delta}{u^2},1]} : W_1^*(s)>u, W_2^*(t)>u\Big\} \\ & + \sum_{L_{2,u}}^{N_u-L_{2,u}} \mathbb{P}\Big\{\exists_{s\in[1-\frac{\Delta}{u^2},1],t\in[t_u-\frac{(k-1)\Delta}{u^2},t_u-\frac{k\Delta}{u^2}]} : W_1^*(s)>u, W_2^*(t)>u\Big\} \\ & +\mathbb{P}\Big\{\exists_{s\in[1-\frac{\Delta}{u^2},1],t\in[1-\frac{\Delta}{u^2},1]} : W_1^*(s)>u, W_2^*(t)>u\Big\} \\ & -(R_1+R_2+R_3+R_4),\end{aligned}$$

where

$$R_1 = \sum_{l=L_{2,u}}^{N_u-L_{2,u}} \sum_{m=l+1}^{N_u-L_{2,u}} \mathbb{P}\{\exists_{s_1\in[\bar{s}_u-\frac{(l-1)\Delta}{u^2},\bar{s}_u-\frac{l\Delta}{u^2}],s_2\in[\bar{s}_u-\frac{(m-1)\Delta}{u^2},\bar{s}_u-\frac{m\Delta}{u^2}],t\in[1-\frac{\Delta}{u^2},1]} : W_1^*(s_1)>u, W_1^*(s_2)>u, W_2^*(t)>u\},$$

$$R_2 = \sum_{l=L_{1,u}}^{N_u-L_{1,u}} \sum_{m=l+1}^{N_u-L_{1,u}} \mathbb{P}\{\exists_{s\in[1-\frac{\Delta}{u^2},1],t_1\in[t_u-\frac{(l-1)\Delta}{u^2},t_u-\frac{l\Delta}{u^2}],t_2\in[t_u-\frac{(m-1)\Delta}{u^2},t_u-\frac{m\Delta}{u^2}]} : W_1^*(s)>u, W_2^*(t_1)>u, W_2^*(t_2)>u\},$$

$$R_3 = \sum_{m=L_{2,u}}^{N_u-L_{2,u}} \sum_{l=\max(m+1,L_{1,u})}^{N_u-\max(m+1,L_{1,u})} \mathbb{P}\Big\{\exists_{s_1\in[1-\frac{\Delta}{u^2},1],s_2\in[\bar{s}_u-\frac{(l-1)\Delta}{u^2},\bar{s}_u-\frac{l\Delta}{u^2}]t_2\in[t_u-\frac{(m-1)\Delta}{u^2},t_u-\frac{m\Delta}{u^2}]} : W_1^*(s_1)>u, W_1^*(s_2)>u, W_2^*(t_2)>u\Big\}.$$

$$R_4 = \sum_{l=L_{1,u}}^{N_u-L_{1,u}} \sum_{m=\max(l+1,L_{2,u})}^{N_u-\max(l+1,L_{2,u})} \mathbb{P}\Big\{\exists_{s_2\in[\bar{s}_u-\frac{(l-1)\Delta}{u^2},\bar{s}_u-\frac{l\Delta}{u^2}],t_1\in[1-\frac{\Delta}{u^2},1],t_2\in[t_u-\frac{(m-1)\Delta}{u^2},t_u-\frac{m\Delta}{u^2}]} : W_1^*(s_2)>u, W_2^*(t_1)>u, W_2^*(t_2)>u\Big\}.$$

Notice that R_1 is the same sum that we calculated in case (ii), while R_2 is a symmetrical variation of it. Since we have that the asymptotics of $\pi_\rho(c_1,c_2;u,u)$ is case (ii) are the same as the asymptotics of S_1, S_2, therefore with (3.15) (if $L_{i,u}=3$) or (3.16) (if $L_{i,u}=-K_u^{(i)}$) we have that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{R_1+R_2}{S_1+S_2}=0.$$

Analogous calculations to (3.15) give

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{R_3+R_4}{S_1+S_2}=0. \tag{3.17}$$

Observe that for any values of $L_{1,u}, L_{2,u}$, we have that S_1 is of the same order as S_2 and hence both S_1 and S_2 contribute to the final asymptotic. We have

$$S_1+S_2 \sim Cu^{-1}\varphi_1(u+c_1,u+c_2),$$

where

$$C = \begin{cases} \sqrt{\frac{3\pi}{2}}\left(e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\left(c_2+\frac{1}{2}c_1\right)+e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\left(c_1+\frac{1}{2}c_2\right)\right), & L_{1,u}=-K_u^{(1)}, L_{2,u}=-K_u^{(2)} \\ \sqrt{\frac{3\pi}{2}}\left(e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\left(c_2+\frac{1}{2}c_1\right)+\frac{1}{2}\right), & L_{1,u}=-K_u^{(1)}, L_{2,u}=3 \\ \sqrt{\frac{3\pi}{2}}\left(\frac{1}{2}+e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\left(c_1+\frac{1}{2}c_2\right)\right), & L_{1,u}=3, L_{2,u}=-K_u^{(2)} \\ \sqrt{\frac{3\pi}{2}}, & L_{1,u}=L_{2,u}=3. \end{cases}$$

Finally notice that from simultaneous ruin results we have $S_3 = O(u^{-2}\varphi_1(u+c_1,u+c_2))$. This completes the proof of case (iv). Note that from the proof we can see that (3.12) holds for $F_u = [1-\frac{\Delta}{u^2},1]\times[t_u - \frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}]\cup[\bar{s}_u - \frac{\log(u)}{u}, 1-\frac{\Delta}{u^2}]\times[1-\frac{\Delta}{u^2},1]$.

Case (v): $\rho < \frac{1}{4a}(1-\sqrt{8a^2+1})$. According to Lemma 3.1 there is exactly one minimizer of $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ on $[0,1]^2$: $(s_u,t_u) = (1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}})$, where for large enough u we have $\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}} < 1$. By (3.11) we have for any $\Delta > 0$

$$\pi_\rho(c_1,c_2;u,au) \;\le\; \sum_{k=1}^{N_u}\sum_{l=-N_u}^{N_u}\mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}}: W_1^*(s)>u, W_2^*(t)>au\Big\}(1+o(1))$$

and, by Bonferroni inequality

$$\begin{aligned}\pi_\rho(c_1,c_2;u,au) \;&\ge\; \mathbb{P}\Big\{\exists(s,t)\in[1-\frac{\Delta}{u^2},1]\times[t_u-\frac{\log u}{u},t_u+\frac{\log u}{u}]:\; W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &\ge\; \sum_{l=-N_u}^{N_u}\mathbb{P}\Big\{\exists_{s\in E^1_{u,1},t\in E^2_{u,l}}: W_1^*(s)>u, W_2^*(t)>au\Big\}\\ &\quad - \sum_{l=-N_u}^{N_u}\sum_{m=l+1}^{N_u}\mathbb{P}\Big\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,l},t_2\in E^2_{u,m}}: W_1^*(s)>u, W_2^*(t_1)>au, W_2^*(t_2)>au\Big\}.\end{aligned} \tag{3.18}$$

The rest of the proof follows by calculations similar to those given in case (ii), with additional use of the asymptotic symmetry of the behaviour of the components in the above summands around point $(1,t_u)$ and the use of Lemma 3.2. Note that from the proof we can see that (3.12) holds for $F_u = [1-\frac{\Delta}{u^2},1]\times[t_u-\frac{\log(u)}{u}, t_u+\frac{\log(u)}{u}]$.

Case (vi): $a=1, \rho<-\frac{1}{2}$. According to Lemma 3.1, there are two minimizers of $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ on $[0,1]^2$: $(s_u,t_u) = (1, \frac{1}{\rho(2\rho-1)+\frac{c_2-\rho c_1}{u}})$ and $(\bar{s}_u,\bar{t}_u) = (\frac{1}{\rho(2\rho-1)+\frac{c_1-\rho c_2}{u}}, 1)$, where for large enough u we have

$$\max\left(\frac{1}{\rho(2\rho-1)+\frac{c_2-\rho c_1}{u}}, \frac{1}{\rho(2\rho-1)+\frac{c_1-\rho c_2}{u}}\right) < 1.$$

Denote

$$F_{1,u} = [1-\frac{\Delta\log u}{u},1]\times[t_u-\frac{\Delta\log u}{u}, t_u+\frac{\Delta\log u}{u}], F_{2,u} = [\bar{s}_u-\frac{\Delta\log u}{u}, \bar{s}_u+\frac{\Delta\log u}{u}]\times[1-\frac{\Delta\log u}{u},1].$$

Using symmetry of the optimizing points and the same idea as in (3.10) we get that as $u \to \infty$

$$\begin{aligned}\pi_\rho(c_1,c_2;u,u) &\le \mathbb{P}\big\{\exists_{(s,t)\in F_{1,u}} : W_1^*(s)>u, W_2^*(t)>u\big\}(1+o(1))\\ &\quad+\mathbb{P}\big\{\exists_{(s,t)\in F_{2,u}} : W_1^*(s)>u, W_2^*(t)>u\big\}(1+o(1)).\end{aligned}$$

Following the same calculations as in case (ii) we have that

$$\begin{aligned}\mathbb{P}\big\{\exists_{(s,t)\in F_{1,u}} : W_1^*(s)>u, W_2^*(t)>u\big\} &\sim \frac{2\sqrt{2\pi}}{t^*\sqrt{\tau_4}}\frac{1-\rho^2t^*}{1-\rho}u^{-1}\varphi_{t_u}(u+c_1,u+c_2t_u)\\ &= \frac{2\sqrt{2\pi}}{t^*\sqrt{\tau_4}}\frac{1-\rho^2t^*}{1-\rho}e^{-\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-\rho)}}u^{-1}\varphi_{t^*}(u+c_1,u+c_2t^*),\end{aligned}$$

where in the last equation we used Lemma 3.2. In a similar way, following case (ii), we get that

$$\begin{aligned}\mathbb{P}\big\{\exists_{(s,t)\in F_{2,u}} : W_1^*(s)>u, W_2^*(t)>u\big\} &\sim \frac{2\sqrt{2\pi}}{t^*\sqrt{\tau_4}}\frac{1-\rho^2t^*}{1-\rho}u^{-1}\varphi_{t_u}(u+c_1t_u,u+c_2)\\ &= \frac{2\sqrt{2\pi}}{t^*\sqrt{\tau_4}}\frac{1-\rho^2t^*}{1-\rho}e^{-\frac{c_1^2-2c_1c_2\rho+c_2^2\rho^2}{2\rho(1-\rho)}}u^{-1}\varphi_{t^*}(u+c_2,u+c_1t^*).\end{aligned}$$

In order to compare the asymptotics above, notice that for

$$\begin{aligned}D_u(c_1,c_2) &:= q^*_{\boldsymbol{a}_u(1,t^*)}(1,t^*)-q^*_{\boldsymbol{a}_u(t^*,1)}(t^*,1)\\ &= (1+\frac{c_1}{u},1+\frac{c_2t^*}{u})\Sigma_{t^*}^{-1}(1+\frac{c_1}{u},1+\frac{c_2t^*}{u})^\top-(1+\frac{c_2}{u},1+\frac{c_1t^*}{u})\Sigma_{t^*}^{-1}(1+\frac{c_2}{u},1+\frac{c_1t^*}{u})^\top\end{aligned}$$

we have for $C=\frac{-\rho(1-t^*)}{1-\rho^2t^*}>0$

$$D_u(c_1,c_2) \sim 2(1,1)\Sigma_{t^*}^{-1}(\frac{c_1-c_2}{u},\frac{(c_2-c_1)t^*}{u})^\top=(c_1-c_2)\frac{C}{u}$$

implying that

(1) For $c_1>c_2 : \varphi_{t^*}(u+c_1,u+c_2t^*)=o(\varphi_{t^*}(u+c_2,u+c_1t^*))$,

(2) For $c_1<c_2 : \varphi_{t^*}(u+c_2,u+c_1t^*)=o(\varphi_{t^*}(u+c_1,u+c_2t^*))$,

(3) For $c_1=c_2 : \varphi_{t^*}(u+c_1,u+c_2t^*)=\varphi_{t^*}(u+c_2,u+c_1t^*)$.

On the other hand as $u\to\infty$

$$\begin{aligned}\pi_\rho(c_1,c_2;u,u) &\ge \mathbb{P}\big\{\exists_{(s,t)\in F_{1,u}} : W_1^*(s)>u, W_2^*(t)>u\big\}+\mathbb{P}\big\{\exists_{(s,t)\in F_{2,u}} : W_1^*(s)>u, W_2^*(t)>u\big\}\\ &\quad-\mathbb{P}\big\{\exists_{(s,t)\in F_{1,u},(s',t')\in F_{2,u}} : W_1^*(s)>u, W_2^*(t)>u, W_1^*(s')>u, W_2^*(t')>u\big\}.\end{aligned}\tag{3.19}$$

Similarly to cases (ii)-(v), (3.19) needs to be shown to be asymptotically negligible. Since calculations are again standard and follow the ideas implemented already in cases (ii) and (v), we omit those calculations. Note that from the proof we can see that (3.12) holds for $F_u=[1-\frac{\Delta}{u^2},1]\times[t_u-\frac{\log(u)}{u},t_u+\frac{\log(u)}{u}]\cup[t_u-\frac{\log(u)}{u},t_u+\frac{\log(u)}{u}]\times[1-\frac{\Delta}{u^2},1]$.

□

A. Appendix

In the Appendix we present some of the proofs of lemmas introduced in the paper.

A.1. **Proof of** (3.10)**.** We follow notation introduced in the proof of Theorem 2.2. Additionally let $H_\varepsilon = ([1-\varepsilon,1]\times[t_u-\varepsilon,\min(t_u+\varepsilon,1)]$. Further denote $Z_u(s,t)=\frac{\boldsymbol{b}_u(s,t)(W_1(s),W_2(t))^\top}{\boldsymbol{b}_u(s,t)\boldsymbol{a}_u(s,t)^\top}$. We have

$$\begin{aligned} Var(Z_u(s,t)) &= \frac{Var(\boldsymbol{b}_u(s,t)(W_1(s),W_2(t))^\top)}{(\boldsymbol{b}_u(s,t)\boldsymbol{a}_u(s,t)^\top)^2}\\ &= \frac{\boldsymbol{a}_u(s,t)\Sigma_{s,t}^{-1}\Sigma_{s,t}\Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t)^\top}{(\boldsymbol{a}_u(s,t)\Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t)^\top)^2}\\ &= \frac{\boldsymbol{a}_u(s,t)\Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t)^\top}{(\boldsymbol{a}_u(s,t)\Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t)^\top)^2}\\ &= \frac{1}{\boldsymbol{a}_u(s,t)\Sigma_{s,t}^{-1}\boldsymbol{a}_u(s,t)^\top}. \end{aligned}$$

Hence Lemma 3.1 for $t_u=\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}$ gives

$$\sigma_u^2 := \sup_{s,t\in[0,1]} Var(Z_u(s,t)) = \frac{1}{\boldsymbol{a}_u(1,t_u)\Sigma_{1,t_u}^{-1}\boldsymbol{a}_u(1,t_u)^\top} > 0.$$

With direct calculations resulting in expansions of the variance in each particular case (see e.g. (3.13)) we have that

$$\sup_{s,t\in[0,1]^2\setminus H_\varepsilon} Var(Z_u(s,t)) < \sigma_u^2 - \bar\tau\epsilon^2$$

implies $\sup_{s,t\in[0,1]^2\setminus H_\varepsilon} Var(Z_u(s,t)) < \frac{\sigma_u^2}{1+\frac{\bar\tau\epsilon^2}{\sigma_u^2}}$ for any $\varepsilon>0$ small enough and some $\bar\tau>0$. Then by Borel-TIS inequality

$$\begin{aligned} \mathbb{P}\{\exists (s,t)\in[0,1]^2\setminus H_\varepsilon : W_1^*(s)>u, W_2^*(t)>au\} &\le \mathbb{P}\{\exists (s,t)\in[0,1]^2\setminus H_\varepsilon : Z_u(s,t)>u\}\\ &\le e^{-r^*\frac{u^2}{2\sigma_u^2}} \end{aligned}$$

for all large u and some $r^*>1$. Recall $F_u=[1-\frac{\log u}{u},1]\times[t_u-\frac{\log u}{u},\min(t_u+\frac{\log u}{u},1)]$. Similarly to above, for all $(s,t)\in H_\varepsilon\setminus F_u$ we have

$$\sigma_u^2 - Var(Z_u(s,t)) \ge \tau\left(\frac{\log u}{u}\right)^2 \tag{A.1}$$

for some $\tau>0$ and all u large. We have

$$\begin{aligned} \mathbb{P}\{\exists (s,t)\in H_\varepsilon\setminus F_u :\ W_1^*(s)>u, W_2^*(t)>au\} &\le \mathbb{P}\{\exists (s,t)\in H_\varepsilon\setminus F_u, s\neq t :\ W_1^*(s)>u, W_2^*(t)>au\}\\ &+ \mathbb{P}\{\exists (s,t)\in H_\varepsilon\setminus F_u, s=t :\ W_1^*(s)>u, W_2^*(t)>au\}\\ &:= P_1+P_2. \end{aligned}$$

Consequently, since Z_u is a Hölder continuous random field and we can choose $\varepsilon > 0$ such that

$$Var(Z_u(s,t)) > 0$$

for all $(s,t) \in H_\varepsilon$, then applying [11][Thm 8.1] for some $c_1, C_1, C_2 > 0$ we have that

$$\begin{aligned} P_1 &\le \mathbb{P}\left\{ \exists (s,t) \in H_\varepsilon \setminus F_u, s \neq t : \frac{Z_u(s,t)}{\sqrt{Var(Z_u(s,t))}} \sqrt{Var(Z_u(s,t))} > u \right\} \\ &\le \mathbb{P}\left\{ \exists (s,t) \in H_\varepsilon \setminus F_u : \frac{Z_u(s,t)}{\sqrt{Var(Z_u(s,t))}} > u \Big/ \sqrt{\sigma_u^2 - \tau \left(\frac{\log u}{u} \right)^2} \right\} \\ &\le C_1 u^{c_1} e^{-\frac{u^2}{2\sigma_u^2 - 2\tau \left(\frac{\log u}{u} \right)^2}} \\ &\le C_1 e^{-C_2 (\log u)^2} e^{-\frac{u^2}{2\sigma_u^2}} = o(\mathbb{P}\{W_1^*(1) > u, W_2^*(t_u) > au\}), \end{aligned}$$

since by Lemma 3.2 and [15][Lem 2] there exists $C, C' > 0, q \in \mathbb{R}$

$$\lim_{u\to\infty} \frac{\mathbb{P}\{W_1^*(1) > u, W_2^*(t^*) > au\}}{u^q e^{-\frac{u^2}{2(\sigma_u)^2}}} = C'.$$

In order to majorize P_2 we first note that

$$\mathbb{P}\{\exists (s,t) \in H_\varepsilon : \ W_1^*(s) > u, W_2^*(t) > au\} \ \ \ge \ \ \mathbb{P}\{W_1^*(1) > u, W_2^*(t_u) > au\}.$$

If $t_u = 1$ by [2][Prop 1.1] for some C positive

$$\begin{aligned} P_2 &\le \mathbb{P}\{\exists s = t, 0 \le s \le 1 - (\log u)/u : \ W_1^*(s) > u, W_2^*(t) > au\} \\ &\le C\mathbb{P}\{W_1^*(1 - (\log u)/u) > u, W_2^*(1 - (\log u)/u) > au\} \\ &= o(\mathbb{P}\{W_1^*(1) > u, W_2^*(1) > au\}), \quad u \to \infty. \end{aligned}$$

Similarly, when $t_u < 1$ by Lemma 3.1 we have that $q^*_{\boldsymbol{a}_u(1,1)}(1,1) < q_{\boldsymbol{a}_u(1,t_u)}(1,t_u)$, hence

$$\begin{aligned} P_2 &\le \mathbb{P}\{\exists s = t, s \in [0,1] : \ W_1^*(s) > u, W_2^*(t) > au\} \\ &\le C^* \mathbb{P}\{W_1^*(1) > u, W_2^*(1) > au\} \\ &= o(\mathbb{P}\{W_1^*(1) > u, W_2^*(t_u) > au\}), \quad u \to \infty, \end{aligned}$$

where the last claim follows from Lemma 3.1 and (3.7). The above implies that

$$\mathbb{P}\big\{\exists (s,t) \in [0,1]^2 \setminus H_\varepsilon : \ W_1^*(s) > u, W_2^*(t) > au\big\} = o(\mathbb{P}\{\exists (s,t) \in H_\varepsilon \setminus F_u : \ W_1^*(s) > u, W_2^*(t) > au\})$$

and

$$\mathbb{P}\{\exists (s,t) \in H_\varepsilon \setminus F_u : \ W_1^*(s) > u, W_2^*(t) > au\} = o(\mathbb{P}\{\exists (s,t) \in F_u : \ W_1^*(s) > u, W_2^*(t) > au\}).$$

Hence (3.10) holds. $\square$

A.2. **Proof of negligibility of (3.15).** Let ϕ denotes the density of $N(0,1)$ random variable. We shall show next that

$$\begin{aligned}\sum_{l=2}^{N_u} K_l &:= \sum_{l=2}^{N_u}\sum_{m=l+1}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,l},t_2\in E^2_{u,m}} : W_1^*(s)>u, W_2^*(t_1)>au, W_2^*(t_2)>au\right\}\\ &= o(u^{-1}\varphi_1(u+c_1,au+c_2)). \qquad \text{(A.2)}\end{aligned}$$

For any $0\le l\le N_u$ using independence of increments of respective Brownian motions we have

$$\begin{aligned}K_l &= \sum_{k=1}^{N_u-l}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s)>u\\ W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\right\}\\
&= \frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\\
&\quad\times\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s)>u\\ W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\middle| W_1(1)=u+c_1-\frac{x}{u}\right\}dx\\
&= \frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\mathbb{P}\left\{\exists_{s\in E^1_{u,1}} : W_1(s)-W_1(1)+c_1(1-s)>\frac{x}{u}\right\}\\
&\quad\times\mathbb{P}\left\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : \begin{array}{c} W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\middle| W_1(1)=u+c_1-\frac{x}{u}\right\}dx\\
&= \frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\mathbb{P}\left\{\exists_{s\in E^1_{u,1}} : W_1(s)-W_1(1)+c_1(1-s)>\frac{x}{u}\right\}\\
&\quad\times\mathbb{P}\left\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\right\}dx,\end{aligned}$$

where $X_{x,u}(t_1,t_2)=(X_{1,x,u}(t_1),X_{2,x,u}(t_2))$ is a bivariate Gaussian process with

$$\mathbb{E}\{X_{x,u}(t_1,t_2)\} = -\begin{pmatrix} -c_2t_1+\rho t_1(c_1-\frac{x}{u})\\ -c_2t_2+\rho t_2(c_1-\frac{x}{u})\end{pmatrix} + \begin{pmatrix} -(a-\rho t_1)u\\ -(a-\rho t_2)u\end{pmatrix}$$

and

$$\Sigma_{X_{x,u}(t_1,t_2)} = \begin{pmatrix} t_1-\rho^2t_1^2 & t_1-\rho^2t_1t_2\\ t_1-\rho^2t_1t_2 & t_2-\rho^2t_2^2\end{pmatrix}.$$

Notice that $X_{x,u}(t_1,t_2)$ is Hölder continuous and let below

$$S_{0,l}=\mathbb{P}\left\{\exists_{t_1\in E^2_{u,l},t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\right\},\quad S_{1,l}=\sum_{k=2}^{N_u-l}\mathbb{P}\left\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\right\},$$

$$S_{2,l} = \mathbb{P}\Big\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2},1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}),t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\Big\},$$

$$S_{3,l} = \mathbb{P}\Big\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2},1-\frac{(l+1)\Delta}{u^2}),t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\Big\}.$$

Observe that to prove (3.15) it is enough to show that as $\Delta\to\infty, u\to\infty$

$$\frac{S_{1,l}+S_{2,l}+S_{3,l}}{S_{0,l}}\to 0$$

uniformly for $l\le N_u$. We have

$$\begin{aligned}\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0\Big\} &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : X_{1,x,u}(t_1)+X_{2,x,u}(t_2)>0\Big\}\\ &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : \frac{X_{1,x,u}(t_1)+X_{2,x,u}(t_2)}{\sigma_{k,u}}>0\Big\},\end{aligned}$$

where $\sigma^2_{k,u} = \max_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}\sigma^2_u(t_1,t_2)$ and $\sigma^2_u(t_1,t_2) := Var(X_{1,x,u}(t_1)+X_{2,x,u}(t_2))$. Since for all $t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}$ we have

$$\lim_{u\to\infty} t_1 = \lim_{u\to\infty} t_2 = 1,$$

then as $u\to\infty$ we have for all $t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}$ that

$$\frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_1} = 3-(2t_1\rho^2+2t_2\rho^2)\sim 3-4\rho^2,\qquad \frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_2} = 1-(2t_2\rho^2+2t_1\rho^2)\sim 1-4\rho^2.$$

Since $\rho^2<\frac14$ then above derivatives are positive for all large u. Hence $\sigma^2_{k,u}(t_1,t_2)$ attains its maximum at $t_1^* = 1-\frac{(l+k)\Delta}{u^2}, t_2^* = 1-\frac{l\Delta}{u^2}$. Consequently

$$\sigma^2_{k,u} = 4-4\rho^2-\frac{1}{u^2}(4l\Delta+3k\Delta-8l\Delta\rho^2-4k\Delta\rho^2)+O(\frac{\Delta^2(k+l)^2}{u^4}).$$

Denote $\mu_u := \mathbb{E}\{X_{1,x,u}(t_1^*)+X_{2,x,u}(t_2^*)\} = 2au+c_2t_1^*+c_2t_2^*-\rho(t_1^*+t_2^*)(u+c_1-\frac{x}{u})$. Using [11][Thm 8.1], there exist constants $C, C_2>0$ such that

$$\begin{aligned}S_{1,l} &\le \sum_{k=2}^{N_u-l}\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}} : \frac{X_{1,x,u}(t_1)+X_{2,x,u}(t_2)}{\sigma_{k,u}}>0\Big\}\\ &\le \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}}e^{-\frac{\mu_u^2}{2\sigma^2_{k,u}}}\\ &= \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}}e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta+3k\Delta-8l\Delta\rho^2-4k\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}}\\ &\le C\frac{\mu_u}{\sigma_{0,u}}e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}}\sum_{k=2}^{N_u-l}e^{-C_2k(\Delta+O(\frac{1}{u^2}))}\\ &\le C\frac{\mu_u}{\sigma_{0,u}}e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}}\frac{e^{-C_2\Delta}}{e^{C_2\Delta}-1}.\end{aligned}\tag{A.3}$$

In the above derivation we used the fact that $4l\Delta + 3k\Delta - 8l\Delta\rho^2 - 4k\Delta\rho^2 > 0$. Similarly we get that

$$S_{2,l} \leq C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}} e^{-C_2\sqrt{\Delta}}. \tag{A.4}$$

Using Lemma 3.4 and Lemma 3.7 we have for any $l = O(\frac{u \log u}{\Delta})$ as $u \to \infty, \Delta \to \infty$

$$\begin{aligned}
\frac{S_{3,l}}{S_{0,l}} &\leq \frac{\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2},1-\frac{(l+1)\Delta}{u^2})} : X_{1,x,u}(t_1) > 0\right\}}{\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2},1-\frac{(l+1)\Delta}{u^2})} : X_{1,x,u}(t_1) > 0\right\}} \\
&\sim \frac{\int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\sqrt{\Delta}]} : W_1(s) - \frac{1-a\rho}{1-\rho^2}s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x}dx \int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\}e^{2ax}dx}{\int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - \frac{1-a\rho}{1-\rho^2}s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x}dx \int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\}e^{2ax}dx} \\
&= \frac{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\}e^{2ax}dx}{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\}e^{2ax}dx} \\
&= \frac{\sqrt{\Delta}}{\Delta} \frac{\int_{\mathbb{R}} \frac{1}{\sqrt{\Delta}}\mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\}e^{2ax}dx}{\int_{\mathbb{R}} \frac{1}{\Delta}\mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\}e^{2ax}dx} \\
&= \frac{\sqrt{\Delta}}{\Delta} > 0. \qquad\qquad (A.5)
\end{aligned}$$

Hence combination of (A.3), (A.4) and (A.5) leads to

$$\frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} \leq \frac{C}{\sqrt{\Delta}} + e^{-C\sqrt{\Delta}} + \frac{e^{-C_2\Delta}}{e^{C_2\Delta} - 1}$$

and therefore

$$\lim_{\Delta\to\infty} \lim_{u\to\infty} \frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} = 0$$

establishing the proof. □

5. Proof of Lemma 3.1

Since it will be needed in the proof of Eq. (3.4) below we consider $a \in [\rho, 1]$ for the derivation of

$$\min_{s,t\in[0,1]\times[0,g]} q_{\boldsymbol{a}_u(ct,t)}(ct, t) = \min_{(z_1,z_2)\in\{1\}\times[0,1]\cup[0,1]\times\{g\}} q_{\boldsymbol{a}_u(z_1,z_2)}(z_1, z_2). \tag{5.1}$$

Supposing that $s \leq t, t = cs, s, t \in [0, 1], c \geq 1$ we have with direct calculations

$$q_{\boldsymbol{a}_u(s,cs)}(s, cs) = \frac{C}{s} + \frac{D_1}{u} + \frac{D_2 s}{u^2},$$

where

$$C = \frac{c - 2a\rho + a^2}{c - \rho^2}, D_1 = \frac{2(ac_2c - c_2\rho c + c_1c - ac_1\rho)}{c - \rho^2}, D_2 = \frac{c(c_1^2 + c_2^2c - 2c_1c_2\rho)}{c - \rho^2}.$$

Since $c > a\rho$ and $a \geq \rho$ implies

$$c - 2a\rho + a^2 = c - a\rho + a(a - \rho) > 0$$

$C > 0$ follows. Hence for $s_2 > s_1$ we have that since $C > 0$

$$\begin{aligned} q_{\boldsymbol{a}_u(s_1,cs_1)}(s_1,cs_1) - q_{\boldsymbol{a}_u(s_2,cs_2)}(s_2,cs_2) &= \frac{C}{s_1} - \frac{C}{s_2} + \frac{D_2 s_1}{u^2} - \frac{D_2 s_2}{u^2} \\ &\geq C(s_2 - s_1) + \frac{D_2}{u^2}(s_1 - s_2) \\ &= (C - \frac{D_2}{u^2})(s_2 - s_1). \end{aligned}$$

Therefore for $u > \sqrt{\frac{|D_2|}{C}}$ we have

$$q_{\boldsymbol{a}_u(s_1,cs_1)}(s_1,cs_1) - q_{\boldsymbol{a}_u(s_2,cs_2)}(s_2,cs_2) > 0$$

for any $1 \geq s_2 > s_1$ and hence $q_{\boldsymbol{a}_u(s,cs)}(s,cs)$ is strictly decreasing in $s \leq t$. Consequently,

$$\min_{0\leq s\leq t\leq 1} q_{\boldsymbol{a}_u(s,cs)}(s,cs) = \min_{z\in[0,1]} q_{\boldsymbol{a}_u(z,1)}(z,1).$$

Similarly, for $s \geq t, s = ct, s,t \in [0,1], c \geq 1$ we have

$$q_{\boldsymbol{a}_u(ct,t)}(ct,t) = \frac{C}{t} + \frac{D_1}{u} + \frac{D_2 t}{u^2},$$

where

$$C = \frac{1 - 2a\rho + a^2 c}{c - \rho^2},\ D_1 = \frac{2(c_1 c - a c_1 c\rho + a c_2 c - c_2\rho)}{c - \rho^2},\ D_2 = \frac{c(c_1^2 c - 2c_1 c_2\rho + c_2^2)}{c - \rho^2}.$$

Since $1 > a\rho$ and $a \geq \rho$, then

$$1 - 2a\rho + a^2 c \geq 1 - 2a\rho + a^2 = 1 - a\rho + a(a - \rho) > 0$$

and therefore $C > 0$. Hence as above for $u > \sqrt{\frac{|D_2|}{C}}$, $q_{\boldsymbol{a}_u(ct,t)}(ct,t)$ is strictly decreasing in $t \leq s$. Consequently, for any $g \in (0,1]$ and any u large

$$\min_{s,t\in[0,1]\times[0,g]} q_{\boldsymbol{a}_u(ct,t)}(ct,t) = \min_{(z_1,z_2)\in\{1\}\times[0,1]\cup[0,1]\times\{g\}} q_{\boldsymbol{a}_u(z_1,z_2)}(z_1,z_2).$$

Next we suppose that $a \in (\rho, 1]$. By the definition of $\boldsymbol{b}(s,t)$ we have that it has both components positive for any $s,t \in (0,1]$ and large enough u and therefore $q^*_{\boldsymbol{a}_u(s,t)}(s,t) = q_{\boldsymbol{a}_u(s,t)}(s,t)$. For any s,t positive such that $\boldsymbol{b}(s,t)$ has positive components we have that $\boldsymbol{a}_u(s,t) = (1 + \frac{c_1 s}{u}, a + \frac{c_2 t}{u})^\top$, which follows from the solution of quadratic programming problem (see [13][Rem 5.1]). Hence

$$q^*_{\boldsymbol{a}_u(s,t)} = \min_{s,t\in[0,1]} q^*_{\boldsymbol{a}_u(s,t)}(s,t) = \min\Big(\min_{z\in[0,1]} q_{\boldsymbol{a}_u(z,1)}(z,1), \min_{z\in[0,1]} q_{\boldsymbol{a}_u(1,z)}(1,z)\Big).$$

Calculating the derivatives we obtain for $z \in [0,1]$

$$\frac{d}{dz} q_{\boldsymbol{a}_u(1,z)}(1,z) = \frac{(\rho^2 - 2a\rho^3 + \frac{2\rho^2 c_1 + 2a\rho^2 c_2 - 2a\rho^3 c_1}{u} + \frac{c^2\rho^2 - 2\rho c_1 c_2 - 2\rho c_2 + c_2^2}{u^2})z^2 + 2a^2\rho^2 z - a^2}{z^2(1-\rho^2 z)^2}$$

and

$$\frac{d}{dz} q_{\boldsymbol{a}_u(z,1)}(z,1) = \frac{(a^2\rho^2 - 2a\rho^3 + \frac{2a\rho^2 c_2 - 2\rho^3 c_2 - 2a\rho c_1 + 2\rho^2 c_1}{u} + \frac{\rho^2 c_2^2 - 2\rho c_1 c_2 + c_1^2}{u^2})z^2 + 2\rho^2 z - 1}{z^2(1-\rho^2 z)^2}.$$

Further notice that

$$\frac{d}{dz}q_{\boldsymbol{a}_u(1,z)}(1,z)=0\Leftrightarrow(z-\frac{a}{\rho-\frac{c_2-\rho c_1}{u}})(z-\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}})=0$$

and

$$\frac{d}{dz}q_{\boldsymbol{a}_u(z,1)}(z,1)=0\Leftrightarrow(z-\frac{1}{a\rho+\frac{c_2\rho-c_1}{u}})(z-\frac{1}{\rho(2\rho-a)-\frac{c_2\rho-c_1}{u}})=0.$$

Hence all potential minimisation points of $q^*_{\boldsymbol{a}_u(s,t)}(s,t)$ are

(1) $(s,t)=(1,1)$,
(2) $(s,t)=(\frac{1}{\rho(2\rho-a)+\frac{c_1-c_2\rho}{u}},1)$,
(3) $(s,t)=(\frac{1}{a\rho-\frac{c_1-c_2\rho}{u}},1)$,
(4) $(s,t)=(1,\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}})$,
(5) $(s,t)=(1,\frac{a}{\rho-\frac{c_2-\rho c_1}{u}})$.

Since points in (3) and (5) do not belong to the $[0,1]\times[0,1]$ for any values of a,ρ and large enough u, then they can be excluded.

(iii) Note that for $\rho>0$ we have

$$\frac{a}{\rho(2a\rho-1)}\le 1\iff a+\rho-2a\rho^2\le 0\iff a-a\rho^2+\rho-\rho^2a\le 0,$$

where the last inequality is clearly false for $\rho<1, a\le 1$ and large enough u. Similarly for $\rho>0$

$$\frac{1}{\rho(2\rho-a)}\le 1\iff 1-2\rho^2+a\rho\le 0\iff 1-\rho^2+a\rho-\rho^2\le 0,$$

where the last inequality is clearly false for $\rho<a\le 1$ and large enough u. Hence for $\rho>0$ points in (2) and in (4) do not belong to $[0,1]\times[0,1]$. Further observe that for $\rho<0$ we have $\min(\frac{a}{\rho(2a\rho-1)},\frac{1}{\rho(2\rho-a)})>0$. Additionally

$$\begin{aligned}\frac{a}{\rho(2a\rho-1)}<1\quad&\iff\quad a+\rho-2a\rho^2<0\iff 1+\frac{\rho}{a}-2\rho^2<0\\ &\Rightarrow\quad 1+\rho a-2\rho^2<0\iff\frac{1}{\rho(2\rho-a)}<1,\end{aligned}$$

where the implication comes from $\rho<0, 0<a\le 1$. Hence the point in (2) belongs to the boundary only if the point in (4) also belongs to the boundary. Additionally notice that

$$\frac{1}{\rho(2\rho-a)}<1\iff 1+a\rho-2\rho^2<0\iff\rho\le\frac{1}{4}(a-\sqrt{a^2+8})<-\frac{1}{2}$$

and hence the point in (2) belongs to the $[0,1]\times[0,1]$ if and only if $\rho<-\frac{1}{2}$. Further we have

$$q^*_{\boldsymbol{a}_u(\frac{1}{\rho(2\rho-a)},1)}(\frac{1}{\rho(2\rho-a)},1)=(2\rho-a)^2+O(\frac{1}{u}),\quad q^*_{\boldsymbol{a}_u(1,\frac{a}{\rho(2a\rho-1)})}(1,\frac{a}{\rho(2a\rho-1)})=(2a\rho-1)^2+O(\frac{1}{u}).$$

We can see that for large enough u

$$\begin{aligned}\rho < -\frac{1}{2} \quad &\Rightarrow \quad 4\rho^2 > 1 \Rightarrow (4\rho^2-1)(1-a^2) > 0\\ &\iff \quad 4\rho^2 + a^2 > 4a^2\rho^2 + 1 \iff 4\rho^2 - 4a\rho + a^2 > 4a^2\rho^2 - 4a\rho + 1\\ &\iff \quad (2\rho - a)^2 > (2a\rho - 1)^2.\end{aligned}$$

Hence if $a < 1$, the point in (2) also cannot be an optimal point. Additionally notice that

$$\begin{aligned} q_{\boldsymbol{a}_u(1,1)}(1,1) > q_{\boldsymbol{a}_u(1,\frac{a}{\rho(2a\rho-1)})}(1,\frac{a}{\rho(2a\rho-1)}) \quad &\iff \quad \frac{1-2a\rho+a^2}{1-\rho^2} > (2a\rho-1)^2\\ &\iff \quad (\rho^2-1)(-2a\rho^2+a+\rho)^2 < 0,\end{aligned}$$

hence the point in (4) is always better than point in (1) (as long as it belongs to $[0,1]\times[0,1]$).

(i)-(ii) For $a = 1, \rho \le -\frac{1}{2}$ the symmetry shows that the symmetrical points (up to the change of roles of c_1 and c_2) are the potential optimal points with the values that converge with u to the same value at $(1, t^*)$. Additionally, for $\rho = -\frac{1}{2}$ the optimal points not always belong to $[0,1]\times[0,1]$, hence the minimum function appears. □

6. Proof of Lemma 3.2

Set $t^* = \frac{a}{\rho(2a\rho-1)} \le 1, t_u = \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}$, where $t_u < 1$ for all large u. We have by direct calculations for all $u \neq 0$

$$\begin{aligned} q^*_{\boldsymbol{a}_u(1,t_u)}(1,t_u) - q_{\boldsymbol{a}_u(1,t^*)}(1,t^*) &= (1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})\Sigma^{-1}_{1,t_u}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\\ &\quad -(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t^*}(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})^\top\\ &= \Big((1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})\Sigma^{-1}_{1,t_u}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\\ &\quad -(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t_u}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\Big)\\ &\quad +\Big((1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t_u}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\\ &\quad -(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t^*}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\Big)\\ &\quad +\Big((1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t^*}(1+\frac{c_1}{u}, a+\frac{c_2 t_u}{u})^\top\\ &\quad -(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})\Sigma^{-1}_{1,t^*}(1+\frac{c_1}{u}, a+\frac{c_2 t^*}{u})^\top\Big)\\ &:= R_1 + R_2 + R_3.\end{aligned}$$

Further we have

$$R_1 = \frac{2ac_2(c_1\rho - c_2)}{u^2\rho(2a\rho-1)}, R_3 = \frac{ac_2(c_1\rho-c_2)(2a\rho^2 u - c_1\rho - 2\rho u + c_2)}{u^2\rho(a\rho-1)(2a\rho^2 u - c_1\rho - \rho u + c_2)}$$

$$
\begin{aligned}
R_2 &= a\frac{4a^2c_1^2\rho^5u - 4a^2c_2^2\rho^3u - 2ac_1^3\rho^4 - 4ac_1^2\rho^4u + 2ac_1^2c_2\rho^3 - 4ac_1c_2\rho^3u + 2ac_1c_2^2\rho^2}{u^2\rho(1-a\rho)(2a\rho^2u - c_1\rho - \rho u + c_2)(2a\rho-1)} \\
&\quad + a\frac{8ac_2^2\rho^2u + c_1^3\rho^3 + c_1^2\rho^3u - 2ac_2^3\rho + 2c_1c_2\rho^2u - 3c_1c_2^2\rho - 3c_2^2\rho u + 2c_2^3}{u^2\rho(1-a\rho)(2a\rho^2u - c_1\rho - \rho u + c_2)(2a\rho-1)}.
\end{aligned}
$$

Combining the above we get that

$$
R_1 + R_2 + R_3 = \frac{a(c_1^2\rho^2 - 2c_1c_2\rho + c_2^2)}{u^2\rho(1-a\rho)}.
$$

□

7. Proof of Lemma 3.4

Let $A_u := \left\{ \begin{array}{c} W_1^*(k_u) = u - \frac{x}{u} \\ W_2^*(l_u) = au - \frac{y}{u} \end{array} \right\}$. For all the cases we can write

$$
\begin{aligned}
&\mathbb{P}\Big\{\exists_{(s,t)\in E_{u,k,l}} : W_1^*(s) > u, W_2^*(t) > au\Big\} \\
&= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{(s,t)\in E} : \begin{array}{c} W_1^*(\frac{s}{u^2} + k_u) > u \\ W_2^*(\frac{t}{u^2} + l_u) > au \end{array} \middle| A_u \right\} \\
&\quad \times u^{-2}\varphi_{k_u,l_u}(u + c_1k_u - \frac{x}{u}, au + c_2l_u - \frac{y}{u})dxdy \\
&= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{(s,t)\in E} : \begin{array}{c} W_1^*(\frac{s}{u^2} + k_u) - W_1(k_u) + W_1(k_u) > u \\ W_2^*(\frac{t}{u^2} + l_u) - W_2(l_u) + W_2(l_u) > au \end{array} \middle| A_u \right\} \\
&\quad \times u^{-2}\varphi_{k_u,l_u}(u + c_1k_u - \frac{x}{u}, au + c_2l_u - \frac{y}{u})dxdy \\
&= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{(s,t)\in E} : \ \chi_{u,k,l}(s,t) > (x,y) \ \middle| A_u \right\} \\
&\quad \times u^{-2}\varphi_{k_u,l_u}(u + c_1k_u - \frac{x}{u}, au + c_2l_u - \frac{y}{u})dxdy.
\end{aligned}
$$

Furthermore, if $k_u \le l_u$, then

$$
\begin{aligned}
&\varphi_{k_u,l_u}(u + c_1k_u - \frac{x}{u}, au + c_2l_u - \frac{y}{u}) \\
&= \frac{1}{2\pi|\Sigma_{k_u,l_u}|} e^{-\frac{1}{2}(u+c_1k_u-\frac{x}{u},au+c_2l_u-\frac{y}{u})\Sigma_{k_u,l_u}^{-1}(u+c_1k_u-\frac{x}{u},au+c_2l_u-\frac{y}{u})^\top} \\
&\sim \frac{1}{2\pi|\Sigma_{1,t_u}|} e^{-\frac{1}{2}(u+c_1k_u,au+c_2l_u)\Sigma_{k_u,l_u}^{-1}(u+c_1k_u,au+c_2l_u)^\top} e^{\frac{l_u - a\rho k_u}{l_uk_u - \rho^2(k_u)^2}x + \frac{a-\rho}{l_u-\rho^2k_u}y} e^{O(u^{-2}(x^2+y^2+xy)+u^{-1}(x+y))} \\
&\sim \frac{1}{2\pi|\Sigma_{1,t_u}|} e^{\lambda_1x+\lambda_2y} e^{-\frac{1}{2}(u+c_1,au+c_2t_u)\Sigma_{1,t_u}^{-1}(u+c_1,au+c_2t_u)^\top} \\
&\quad \times e^{\frac{1}{2}(u+c_1,au+c_2t_u)\Sigma_{1,t_u}^{-1}(u+c_1,au+c_2t_u)^\top - \frac{1}{2}(u+c_1k_u,au+c_2l_u)\Sigma_{k_u,l_u}^{-1}(u+c_1k_u,au+c_2l_u)^\top} \\
&\sim \varphi_{t_u}(u + c_1, au + c_2t_u)e^{\lambda_1x+\lambda_2y} \\
&\quad \times e^{-\frac{1}{2}(u+c_1k_u,au+c_2l_u)\Sigma_{k_u,l_u}^{-1}(u+c_1k_u,au+c_2l_u)^\top + \frac{1}{2}(u+c_1,au+c_2t_u)\Sigma_{1,t_u}^{-1}(u+c_1,au+c_2t_u)^\top}
\end{aligned}
$$

$$= \varphi_{t_u}(u+c_1, au+c_2t_u)e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,t_u)}(1,t_u))}e^{\lambda_1 x+\lambda_2 y}.$$

Using Lemma 3.2 from [16] we have

$$\begin{aligned}&\varphi_{k_u,l_u}(u+c_1k_u-\frac{x}{u}, au+c_2l_u-\frac{y}{u})\\ &= e^{-\mathbf{1}(t_u<1)\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1, au+c_2t^*)e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,t_u)}(1,t_u))}e^{\lambda_1 x+\lambda_2 y}.\end{aligned}$$

Similarly, for $k_u > l_u$ we have

$$\begin{aligned}&\varphi_{k_u,l_u}(u+c_1k_u-\frac{x}{u}, au+c_2l_u-\frac{y}{u})\\ &\sim \varphi_{t^*}(u+c_1, au+c_2t^*)e^{-\mathbf{1}(s_u<1)\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u,l_u)-q_{\boldsymbol{a}_u(1,t_u)}(1,t_u))}e^{\lambda_1 x+\lambda_2 y}.\end{aligned}$$

Next we investigate

$$I_u = \int_{\mathbb{R}^2}\mathbb{P}\{\exists_{(s,t)\in E}: \chi^*_{u,k,l,x,y}(s,t) > (x,y)\}e^{\lambda_1 x+\lambda_2 y}dxdy,$$

where $\chi^*_{u,k,l,x,y}(s,t) = (\chi^*_{1,u,k,l,x,y}(s), \chi^*_{2,u,k,l,x,y}(t)) := (\chi_{u,k,l}(s,t)|A_u),\ s,t\in[-\Delta,0]$. It appears that the play between k_u and l_u influences the next steps of argumentation, hence the rest of the proof is divided into three cases, $k_u < l_u, k_u = l_u, k_u > l_u$.

(i) Suppose that $k_u = l_u$. Then $\mathbb{E}\{\chi^*_{u,k,l,x,y}(s,t)\} = -\frac{1}{uk_u}\begin{pmatrix} s(u+c_1k_u-\frac{x}{u}) \\ t(au+c_2k_u-\frac{y}{u})\end{pmatrix}$ and the covariance matrix of $\chi^*_{u,k,l,x,y}(s,t)$ is equal to

$$\begin{aligned}\Sigma_{(\chi^*_{u,k,l,x,y}(s,t))} &= \begin{pmatrix} s & \rho\min(s,t) \\ \rho\min(s,t) & t\end{pmatrix} - u^{-2}\begin{pmatrix} s & \rho s \\ \rho t & t\end{pmatrix}\begin{pmatrix} k_u & \rho k_u \\ \rho k_u & l_u\end{pmatrix}^{-1}\begin{pmatrix} s & \rho t \\ \rho s & t\end{pmatrix}\\ &= \begin{pmatrix} s & \rho\min(s,t) \\ \rho\min(s,t) & t\end{pmatrix} - O\left(\frac{\log u}{u}\right)\begin{pmatrix} s^2 & \rho^2 st \\ (\rho\min(s,t))^2 & t^2\end{pmatrix}, s,t\in[0,\Delta].\end{aligned}$$

Additionally since

$$\chi^*_{u,k,l,x,y}(s_1,t_1) - \chi^*_{u,k,l,x,y}(s_2,t_2) \overset{d}{=} u\Big(W_1(\frac{s_1}{u^2}+k_u) - W_1(\frac{s_2}{u^2}+k_u), W_2(\frac{t_1}{u^2}+l_u) - W_2(\frac{t_2}{u^2}+l_u)|A_u\Big),$$

then $\Sigma_{(\chi^*_{u,k,l,x,y}(s_1,t_1)-\chi^*_{u,k,l,x,y}(s_2,t_2))} = \Sigma_{\chi^*_{u,k,l,x,y}(|s_1-s_2|,|t_1-t_2|)}$. Using the above and the continuous mapping theorem we get, as $u\to\infty$

$$I_u \sim \int_{\mathbb{R}^2}\mathbb{P}\left\{\exists_{s,t\in[0,\Delta]}: \begin{array}{c} W_1(s)-s > x \\ W_2(t)-at > y\end{array}\right\}e^{\lambda_1 x+\lambda_2 y}dxdy.$$

In order to justify the application of dominated convergence theorem we show the finitness of (3.8) . With $\lambda_1, \lambda_2 > 0$ we get for sufficiently large u

$$
\begin{aligned}
I_u &= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists_{(s,t)\in E} : \begin{array}{c} \chi^*_{1,u,k,l,x,y}(s) > x \\ \chi^*_{2,u,k,l,x,y}(t) > y \end{array} \right\} e^{\lambda_1 x + \lambda_2 y} dxdy \\
&\le \int_{\mathbb{R}_-}\int_{\mathbb{R}_-} e^{\lambda_1 x+\lambda_2 y} dxdy + \int_{\mathbb{R}_-}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \chi^*_{1,u,k,l,x,y}(s) > x\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,k,l,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{(s,t)\in[0,\Delta]} : \chi^*_{1,u,k,l,x,y}(s) + \chi^*_{2,u,k,l,x,y}(t) > x+y\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\le \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2}\int_{\mathbb{R}_+} C_1 e^{-C_2 x^2} e^{\lambda_1 x} dx \qquad (7.1) \\
&\quad + \frac{1}{\lambda_1}\int_{\mathbb{R}_+} C_1 e^{-C_2 y^2} e^{\lambda_2 y} dy + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} C_1 e^{-C_2(x+y)^2} e^{\lambda_1 x+\lambda_2 y} dxdy < \infty,
\end{aligned}
$$

where (7.1) follows from [11][Thm 8.1] with some constants $C_1, C_2 > 0$.

(ii) Suppose that $k_u < l_u$. Then the increments $W_1(s+k_u u^2) - W_1(k_u u^2), W_2(t+l_u u^2) - W_2(l_u u^2)$ are mutually independent. Hence

$$
I_u = \int_{\mathbb{R}^2} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \chi^*_{1,u,x,y}(s) > x\}\mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy,
$$

where $\chi^*_{1,u,x,y}(s) := \chi_{1,u,k}(s)|A_u$ is a Gaussian process with

$$
\mathbb{E}\{\chi^*_{1,u,x,y}(s)\} = \frac{s}{uk_u}(u + c_1 - \frac{x}{u}) - c_1\frac{s}{u},
$$

$$
Var\left(\chi^*_{1,u,x,y}(s)\right) = s - s^2\frac{l_u - \rho^2 k_u}{u^2(k_u l_u - \rho^2 k_u^2)} = s - \frac{s^2}{u^2 k_u}
$$

and $\chi^*_{2,u,x,y}(t) := \chi_{2,u,l}(t)|A_u$ is a Gaussian process with

$$
\mathbb{E}\{\chi^*_{2,u,x,y}(t)\} = \frac{1}{u(l_u k_u - \rho^2 k_u^2)}(k_u t(au + c_2 - \frac{y}{u}) - \rho k_u t(u + c_1 - \frac{x}{u})) - c_2\frac{t}{u},
$$

$$
Var\left(\chi^*_{2,u,x,y}(t)\right) = t - t^2\frac{k_u}{u^2(k_u l_u - \rho^2 k_u^2)} = t - \frac{t^2}{u^2(l_u - \rho^2 k_u)}.
$$

Moreover, for each $0 \ge s > t \ge -\Delta$, $\chi^*_{1,u,x,y}(s) - \chi^*_{1,u,x,y}(t)$ is normally distributed with

$$
Var\left(\chi^*_{1,u,x,y}(s) - \chi^*_{1,u,x,y}(t)\right) = (s-t) - \frac{(s-t)^2}{u^2 k_u}
$$

while $\chi^*_{2,u,x,y}(s) - \chi^*_{2,u,x,y}(t)$ is normally distributed with

$$
Var\left(\chi^*_{2,u,x,y}(s) - \chi^*_{2,u,x,y}(t)\right) = (s-t) - \frac{(s-t)^2}{u^2(l_u - \rho^2 k_u)}.
$$

Hence, using that $Var\left(\chi^*_{i,u,x,y}(s) - \chi^*_{i,u,x,y}(t)\right) \le 2|s-t|$ for all large enough u, we conclude that $\chi^*_{1,u,x,y}(s)$ weakly converges to $W_1(s) - s$ and $\chi^*_{2,u,x,y}(t)$ weakly converges to $W_2(t) - \frac{a-\rho}{t^*-\rho^2}t$ in $C[0,\Delta]$.

Next we prove the finitness of (3.8) to justify the application of dominated convergence theorem. We have

$$
\begin{aligned}
I_u &\leq \int_{\mathbb{R}^2} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \chi^*_{1,u,x,y}(s) > x\}\mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\leq \int_{\mathbb{R}_-}\int_{\mathbb{R}_-} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_-}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \chi^*_{1,u,x,y}(s) > x\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \chi^*_{1,u,x,y}(s) > x\}\mathbb{P}\{\exists_{t\in[0,\Delta]} : \chi^*_{2,u,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy.
\end{aligned}
$$

Since $Var(\chi^*_{i,u,x,y}(s) - \chi^*_{i,u,x,y}(t)) \leq 2|s-t|$ for all large enough u by [11][Thm 8.1]

$$
\begin{aligned}
I_u &\leq \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2}\int_{\mathbb{R}_+} C_1 e^{-C_2 x^2} e^{\lambda_1 x} dx \\
&\quad + \frac{1}{\lambda_1}\int_{\mathbb{R}_+} C_1 e^{-C_2 y^2} e^{\lambda_2 y} dy + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} C_1 e^{-C_2(x^2+y^2)} e^{\lambda_1 x+\lambda_2 y} dxdy < \infty.
\end{aligned}
$$

From the above it holds that (3.8) is finite. Combining it with the weak convergence proven above and the dominated convergence theorem, we obtain that

$$
\lim_{u\to\infty} I_u = \int_{\mathbb{R}^2} \mathbb{P}\{\exists_{s\in[0,\Delta]} : W_1(s) - s > x\}\mathbb{P}\left\{\exists_{t\in[0,\Delta]} : W_2(t) - \frac{a-\rho}{t^*-\rho^2} t > y\right\} e^{\lambda_1 x+\lambda_2 y} dxdy.
$$

(iii) Suppose that $k_u > l_u$. Then the increments $W_1(s+k_u u^2) - W_1(k_u u^2), W_2(t+l_u u^2) - W_2(l_u u^2)$ are mutually independent. Hence we have

$$
I_u = \int_{\mathbb{R}^2} \mathbb{P}\{\exists_{s\in[0,\Delta]} : \xi^*_{1,u,x,y}(s) > x\}\mathbb{P}\{\exists_{t\in[0,\Delta]} : \xi^*_{2,u,x,y}(t) > y\} e^{\lambda_1 x+\lambda_2 y} dxdy,
$$

where $\xi^*_{1,u,x,y}(s) := \chi_{1,u,k}(s)|A_u$ is a Gaussian process with

$$
\mathbb{E}\{\xi^*_{1,u,x,y}(s)\} = \frac{1}{u(l_u k_u - \rho^2 l_u^2)}(s l_u(u + c_1 - \frac{x}{u}) - \rho s l_u(au + c_2 - \frac{y}{u})) - c_1\frac{s}{u},
$$

$$
Var\left(\xi^*_{1,u,x,y}(s)\right) = s - s^2\frac{l_u}{u^2(k_u l_u - \rho^2 l_u^2)} = s - \frac{s^2}{u^2(k_u - \rho^2 l_u)}
$$

and $\xi^*_{2,u,x,y}(t) := \chi_{2,u,l}(t)|A_u$ is a Gaussian process with

$$
\mathbb{E}\{\xi^*_{2,u,x,y}(t)\} = \frac{t}{u l_u}(au + c_2 - \frac{y}{u}) - c_2\frac{t}{u},
$$

$$
Var\left(\xi^*_{2,u,x,y}(t)\right) = t - t^2\frac{k_u - \rho^2 l_u}{u^2(k_u l_u - \rho^2 l_u^2)} = t - \frac{t^2}{u^2 l_u}.
$$

Moreover, for each $0 \geq s > t \geq -\Delta$, $\xi^*_{1,u,x,y}(s) - \xi^*_{1,u,x,y}(t)$ is normally distributed with

$$
Var\left(\xi^*_{1,u,x,y}(s) - \xi^*_{1,u,x,y}(t)\right) = (s-t) - \frac{(s-t)^2}{u^2(k_u - \rho^2 l_u)}
$$

and $\xi^*_{2,u,x,y}(s) - \xi^*_{2,u,x,y}(t)$ is normally distributed with

$$Var\left(\xi^*_{2,u,x,y}(s) - \xi^*_{2,u,x,y}(t)\right) = (s-t) - \frac{(s-t)^2}{u^2 l_u}.$$

Hence, using that $Var\left(\xi^*_{i,u,x,y}(s) - \xi^*_{i,u,x,y}(t)\right) \le 2|s-t|$ for all u large enough, $\xi^*_{1,u,x,y}(s)$ weakly converges to $W_1(s) - \frac{t^*(1-a\rho)}{t^*-\rho^2(t^*)^2}s$ and $\xi^*_{2,u,x,y}(t)$ weakly converges to $W_2(t) - \frac{a}{t^*}t$ in $C[0,\Delta]$. This leads to

$$\lim_{u\to\infty} I_u = \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - \frac{1-a\rho}{1-\rho^2 t^*}s > x\right\}\mathbb{P}\left\{\exists_{t\in[0,\Delta]} : W_2(t) - \frac{a}{t^*}t > y\right\}e^{\lambda_1 x + \lambda_2 y}dxdy.$$

The finitness of (3.8) and the application of the dominated convergence theorem can be proven identically as in case (ii). This completes the proof.

8. Proof of Lemma 3.5

Ad i). The proof follows straightforwardly from the fact that

$$\mathbb{P}\{\sup_{t\in[0,\infty)} (B(t) - bt) > x\} = \min(1, e^{-2bx})$$

for $x \in \mathbb{R}$.

Ad ii). Note that, by self-similarity of Brownian motion and the change of variables $y = 2bx$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,T]} (B(t) - bt) > x\}e^{2bx}dx &= \int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,2b^2T]} \left(B\left(\frac{t}{2b^2}\right) - \frac{t}{2b}\right) > x\}e^{2bx}dx \\
&= \int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,2b^2T]} (\sqrt{2}B(t) - t) > 2bx\}e^{2bx}dx \\
&= \frac{1}{2b}\int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,2b^2T]} (\sqrt{2}B(t) - t) > y\}e^{y}dy.
\end{aligned}$$

Hence, using that

$$\lim_{T\to\infty} \frac{1}{T}\int_{\mathbb{R}} \mathbb{P}\{\sup_{t\in[0,T]} (\sqrt{2}B(t) - t) > y\}e^{y}dy = 1$$

(see e.g. [11]) we complete the proof. □

9. Proof of Lemma 3.6

With $a \in (\max(0,\rho), 1]$ define

$$\boldsymbol{\lambda} = \Sigma^{-1}\boldsymbol{a} > \mathbf{0}, \quad \lambda_1 = \frac{1-a\rho}{1-\rho^2}, \quad \lambda_2 = \frac{a-\rho}{1-\rho^2}.$$

Since λ_1, λ_2 are positive, then for any $\Delta > 0$

$$\int_{x_1\le 0, x_2\le 0} v(\boldsymbol{x})\, dx \in (0,\infty),$$

where

$$v(\boldsymbol{x}) = \mathbb{P}\big\{\exists_{\boldsymbol{s}\in[0,\Delta]^2} : \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{q}\cdot\boldsymbol{s}] > \boldsymbol{x}\big\} e^{\boldsymbol{a}^\top \Sigma^{-1}\boldsymbol{x}}.$$

If one of the coordinates of $\boldsymbol{x}$ is negative, then the integral reduces to one-dimensional case and it follows easily from (1.2) that this integral is also bounded for any $\Delta > 0$. The finiteness of $I(\Delta, \boldsymbol{q})$ for $\Delta > 0$ follows if we show further that

$$\int_{x_1>0, x_2>0} v(\boldsymbol{x})\, d\boldsymbol{x}$$

is finite, which can be established by using the fact that

$$\log(\mathbb{P}\big\{\exists_{\boldsymbol{s}\in[0,\Delta]^2} : \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{q}\cdot\boldsymbol{s}] > \boldsymbol{x}\big\}) \le -c\boldsymbol{x}^\top\boldsymbol{x}$$

for some $c > 0$, which comes from [12][Thm 1].

Now we prove the second claim. Note that for positive μ_1, μ_2 and (X, Y) a bivariate random vector with finite moment generating function

$$\int_{\mathbb{R}^2} \mathbb{P}\{X > s, Y > t\} e^{\mu_1 s + \mu_2 t} ds dt \quad = \quad \frac{1}{\mu_1\mu_2}\mathbb{E}\{e^{\mu_1 X + \mu_2 Y}\}.$$

Next, for $\Delta = n \in \mathbb{N}, \boldsymbol{\lambda}$ defined previously

$$\begin{aligned}
\int_{\mathbb{R}^2} v(\boldsymbol{x})\, d\boldsymbol{x} \quad &\le \quad \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \int_{\mathbb{R}^2} \mathbb{P}\big\{\exists_{\boldsymbol{s}\in[i,i+1]\times[j,j+1]} \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{a}\cdot\boldsymbol{s}] > \boldsymbol{x}\big\} e^{\boldsymbol{a}^\top\Sigma^{-1}\boldsymbol{x}} d\boldsymbol{x} \\
&= \quad \frac{1}{\lambda_1\lambda_2}\sum_{i=0}^{n-1}\sum_{j=0, j\ne i}^{n-1} \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{M}(i,j)}\} + \frac{1}{\lambda_1\lambda_2}\sum_{i=0}^{n-1}\mathbb{E}\{e^{\boldsymbol{\lambda}^\top\boldsymbol{M}(i,i)}\},
\end{aligned}$$

where (below supremum is taken component-wise)

$$\begin{aligned}
\boldsymbol{M}(i,i) \quad &= \quad \sup_{\boldsymbol{s}\in[i,i+1]\times[i,i+1]} \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{a}\cdot\boldsymbol{s} \\
&= \quad \sup_{\boldsymbol{s}\in[i,i+1]\times[i,i+1]} \boldsymbol{W}(\boldsymbol{s}) - (W_1(i), W_2(i))^\top - \boldsymbol{a}\cdot(\boldsymbol{s} - (i,i)^\top + (i,i)^\top) + (W_1(i), W_2(i))^\top \\
&\stackrel{d}{=} \quad \sup_{\boldsymbol{s}\in[0,1]\times[0,1]} \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{a}\cdot\boldsymbol{s} - \boldsymbol{a}\cdot(i,i)^\top + (W_1^*(i), W_2^*(i))^\top \\
&=: \quad \boldsymbol{Q} - \boldsymbol{a}\cdot(i,i)^\top + (W_1^*(i), W_2^*(i))^\top
\end{aligned}$$

and for $j > i$ (case $i > j$ yields similar result)

$$\begin{aligned}
\boldsymbol{M}(i,j) \quad &= \quad \sup_{\boldsymbol{s}\in[i,i+1]\times[j,j+1]} \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{a}\cdot\boldsymbol{s} \\
&= \quad \sup_{(s,t)\in[i,i+1]\times[j,j+1]} (W_1(s) - W_1(i), W_2(i+1) - W_2(i))^\top + (0, W_2(t) - W_2(j))^\top + \\
&\quad\quad +(0, W_2(j) - W_2(i+1))^\top - \boldsymbol{a}\cdot\boldsymbol{(}s - i, t - aj)^\top - \boldsymbol{a}\cdot\boldsymbol{(}i, aj)^\top + (W_1(i), W_2(i))^\top \\
&\stackrel{d}{=} \quad \sup_{\boldsymbol{s}\in[0,1]\times[0,1]} \boldsymbol{W}(\boldsymbol{s}) - \boldsymbol{a}\cdot\boldsymbol{s} + (W_1^*(i), W_2^*(j))^\top - (i, aj)^\top
\end{aligned}$$

$$\stackrel{d}{=} \quad \boldsymbol{Q} + (W_1^*(i), W_2^*(j))^\top - (i, aj)^\top,$$

where (W_1^*, W_2^*) is an independent copy of (W_1, W_2) and $\stackrel{d}{=}$ stands for equality in law. Observe that since $\lambda_1 + \lambda_2\rho = 1$ we have

$$\boldsymbol{\lambda}^\top (W_1(i), W_2(i))^\top = (\lambda_1 + \lambda_2\rho) B_1(i) + \lambda_2\rho^* B_2(i) = B_1(i) + \lambda_2\rho^* B_2(i)$$

and $\lambda_1 > \frac{1}{2}$ and $\lambda_2 < 2a$, then for some C positive

$$\begin{aligned}
\log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{M}(i,i)}\} &= \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} + \frac{1}{2} i + \frac{\lambda_2^2}{2}(1-\rho^2) i - \lambda_1 i - \lambda_2 a i \\
&= \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} - i\left(\lambda_1 - \frac{1}{2}\right) - i\left(\lambda_2 a - \frac{\lambda_2^2}{2}(1-\rho^2)\right) \\
&\leq \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} - Ci
\end{aligned}$$

and for $j > i$ for some $C_1 > 0, C_2$

$$\begin{aligned}
\log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{M}(i,j)}\} &= \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} + \frac{1}{2} i + \frac{\lambda_2^2}{2}(1-\rho^2) i + \frac{\lambda_2^2}{2}(j-i) - \lambda_1 i - \lambda_2 a j \\
&\leq \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} - i\left(\lambda_1 - \frac{1}{2}\right) - j\left(\lambda_2 a - \frac{\lambda_2^2}{2}\right) \\
&\leq \log \mathbb{E}\{e^{\boldsymbol{\lambda}^\top \boldsymbol{Q}}\} - C_1(i+j),
\end{aligned}$$

hence the claim follows. □

10. Proof of Lemma 3.7

Let $\Delta > 0$. Using Riemann approximation of the integral with step $\frac{\sqrt{C}\Delta}{u}$ and monotonicity of the function $e^{-\frac{x^2}{2}}$ we have for large enough u

$$Q_{1, \frac{u \log(u)}{\Delta}} \sim \int_0^{\log u} e^{-\frac{x^2}{2}} dx \to \int_0^\infty e^{-\frac{x^2}{2}} dx, u \to \infty.$$

With

$$\int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{\sqrt{\pi}}{2} < \infty$$

we have that

$$\lim_{u \to \infty} Q_{1, \frac{u \log(u)}{\Delta}} = \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{\sqrt{\pi}}{2}$$

Using the same arguments we have that

$$Q_{-\frac{cu}{\Delta}, \frac{u \log(u)}{\Delta}} \sim \int_{-c}^\infty e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}\Phi(c)$$

and

$$Q_{-\frac{u \log(u)}{\Delta}, \frac{u \log(u)}{\Delta}} \sim \int_{-\infty}^\infty e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$

□

11. Proof of (3.4)

In view of (5.1) we have with $g = h_u$

$$\min_{s\in[0,1],t\in[0,h_u]} q_{\boldsymbol{a}_u(s,t)}(s,t) = \min_{(z_1,z_2)\in\{1\}\times[0,1]\cup[0,1]\times\{h_u\}} q_{\boldsymbol{a}_u(s,t)}(z_1,z_2).$$

Furthermore

$$\frac{\partial}{\partial s} q_{\boldsymbol{a}_u(s,t)}(s,h_u) < 0, s \le 1, \quad \frac{\partial}{\partial t} q_{\boldsymbol{a}_u(s,t)}(1,t) < 0, t < h_u$$

implying that

$$\min_{s\in[0,1],t\in[0,h_u]} q_{\boldsymbol{a}(s,t)}(s,t) = q_{\boldsymbol{a}(1,h_u)}(1,h_u).$$

The random field

$$Z_u(s,t) = \frac{b_{1,u}(s,t)W_1(s) + b_{2,u}(s,t)W_2(t)}{b_{1,u}(s,t)(1+\frac{c_1 s}{u}) + b_{2,u}(s,t)(\rho + \frac{c_2 t}{u})}, s,t \in [0,1]$$

has variance function equal to $1/q_{\boldsymbol{a}_u(s,t)}(s,t)$. For large enough u the first component of $\boldsymbol{b}_u(s,t)$ is positive and the second component is equal to zero for all $s \le t$ and is positive for $s > t$. Consequently,

$$\sup_{s\in[0,1],t\in[0,h_u]} Var(Z_u(s,t)) = \sup_{s\in[0,1],t\in[0,h_u]} \frac{1}{q_{\boldsymbol{a}_u(s,t)}(s,t)} = \frac{1}{\inf_{s\in[0,1],t\in[0,h_u]} q_{\boldsymbol{a}_u(s,t)}(s,t)} = \frac{1}{q_{\boldsymbol{a}_u(1,h_u)}(1,h_u)}.$$

Next, for $h_u = 1 - 1/\sqrt{u}$

$$1 - q_{\boldsymbol{a}_u(1,h_u)}(1,h_u) \quad = \quad 1 - \frac{h_u - 2\rho^2(h_u) + \rho^2}{h_u - \rho^2 h_u^2} \sim -\frac{1}{u}\frac{\rho^2}{1-\rho^2}(1+o(1))$$

establishing the proof. □

12. Proof of negligibility of (3.16)

The proof will follow the path of the proof of negligibility of (3.15) . For any $-K_u^{(1)} \le l \le N_u$ using independence of increments of respective Brownian motions we have

$$\sum_{k=1}^{N_u - l} \mathbb{P}\left\{ \exists_{s\in E^1_{u,1}, t_1 \in E^2_{u,k+l}, t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s) > u \\ W_2^*(t_1) > au \\ W_2^*(t_2) > au \end{array} \right\}$$

$$= \frac{1}{u}\sum_{k=1}^{N_u - l}\int_{\mathbb{R}} \phi(u + c_1 - \frac{x}{u})\mathbb{P}\Big\{\exists_{s\in E^1_{u,1}} : W_1(s) - W_1(1) + c_1(1-s) > \frac{x}{u}\Big\}$$

$$\times \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l}, t_2 \in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\Big\} dx,$$

where $X_{x,u}(t_1,t_2) = (X_{1,x,u}(t_1), X_{2,x,u}(t_2))$ is a bivariate Gaussian process with

$$\mathbb{E}\{X_{x,u}(t_1,t_2)\} = -\begin{pmatrix} -c_2t_1 + \rho t_1(c_1 - \frac{x}{u}) \\ -c_2t_2 + \rho t_2(c_1 - \frac{x}{u}) \end{pmatrix} + \begin{pmatrix} -(a-\rho t_1)u \\ -(a-\rho t_2)u \end{pmatrix}$$

and

$$\Sigma_{X_{x,u}(t_1,t_2)} = \begin{pmatrix} t_1 - \rho^2 t_1^2 & t_1 - \rho^2 t_1 t_2 \\ t_1 - \rho^2 t_1 t_2 & t_2 - \rho^2 t_2^2 \end{pmatrix}.$$

Notice that $X_{x,u}(t_1,t_2)$ is Hölder continuous and let below

$$S_{0,l} = \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,l}, t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\Big\}, \quad S_{1,l} = \sum_{k=2}^{N_u - l} \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\Big\},$$

$$S_{2,l} = \mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2}, 1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}), t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\right\},$$

$$S_{3,l} = \mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}, 1-\frac{(l+1)\Delta}{u^2}), t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\right\}.$$

Observe that for (3.16) to be negligible it is enough to show that for all $-K_u^{(1)} \le l \le N_u$ as $u\to\infty, \Delta\to\infty$

$$\frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} \to 0.$$

We have

$$\begin{aligned} \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2) > 0\Big\} &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}} : X_{1,x,u}(t_1) + X_{2,x,u}(t_2) > 0\Big\} \\ &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}} : \frac{X_{1,x,u}(t_1) + X_{2,x,u}(t_2)}{\sigma_{k,u}} > 0\Big\}, \end{aligned}$$

where $\sigma^2_{k,u} = \max_{t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}} \sigma^2_u(t_1,t_2)$ and $\sigma^2_u(t_1,t_2) := Var(X_{1,x,u}(t_1) + X_{2,x,u}(t_2))$. Since for all $t_1 \in E^2_{u,k+l}, t_2\in E^2_{u,l}$ we have

$$\lim_{u\to\infty} t_1 = \lim_{u\to\infty} t_2 = 1,$$

then as $u\to\infty$ we have for all $t_1\in E^2_{u,k+l}, t_2\in E^2_{u,l}$ that

$$\frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_1} = 3 - (2t_1\rho^2 + 2t_2\rho^2) \sim 3 - 4\rho^2, \quad \frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_2} = 1 - (2t_2\rho^2 + 2t_1\rho^2) \sim 1 - 4\rho^2.$$

Since $\rho^2 < \frac{1}{4}$ then above derivatives are positive for all large u. Hence $\sigma^2_{k,u}(t_1,t_2)$ attains its maximum at $t^*_{1,u} = t_u - \frac{(l+k)\Delta}{u^2}, t^*_{2,u} = t_u - \frac{l\Delta}{u^2}$. Consequently

$$\sigma^2_{k,u} = 4 - 4\rho^2 - \frac{1}{u^2}(4l\Delta + 3k\Delta - 8l\Delta\rho^2 - 4k\Delta\rho^2) + O(\frac{\Delta^2(k+l)^2}{u^4}).$$

Denote $\mu_u := \mathbb{E}\{X_{1,x,u}(t_{1,u}^*) + X_{2,x,u}(t_{2,u}^*)\} = 2au + c_2 t_{1,u}^* + c_2 t_{2,u}^* - \rho(t_{1,u}^* + t_{2,u}^*)(u + c_1 - \frac{x}{u})$. Using [11][Thm 8.1], there exist constants $C, C_2 > 0$ such that

$$\begin{aligned}
S_{1,l} &\le \sum_{k=2}^{N_u-l} \mathbb{P}\left\{\exists_{t_1\in E_{u,k+l}^2, t_2\in E_{u,l}^2} : \frac{X_{1,x,u}(t_1)+X_{2,x,u}(t_2)}{\sigma_{k,u}} > 0\right\}\\
&\le \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu_u^2}{2\sigma_{k,u}^2}}\\
&= \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta+3k\Delta-8l\Delta\rho^2-4k\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}}\\
&\le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}} \sum_{k=2}^{N_u-l} e^{-C_2 k(\Delta+O(\frac{1}{u^2}))}\\
&\le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}} \frac{e^{-C_2\Delta}}{e^{C_2\Delta}-1}.
\end{aligned} \tag{12.1}$$

In the above we used the fact that $4l\Delta + 3k\Delta - 8l\Delta\rho^2 - 4k\Delta\rho^2 > 0$. Similarly we get that

$$S_{2,l} \le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{1}{u^2}(4l\Delta-8l\Delta\rho^2)+O(\frac{1}{u^4}))}{2((4-4\rho^2)^2+O(\frac{1}{u^4}))}} e^{-C_2\sqrt{\Delta}}. \tag{12.2}$$

Using Lemma 3.4 from [16] and Lemma 3.7 from [16] we have as $u \to \infty$

$$\begin{aligned}
\frac{S_{3,l}}{S_{0,l}} &\le \frac{\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}, 1-\frac{(l+1)\Delta}{u^2})} : X_{1,x,u}(t_1) > 0\right\}}{\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2}, 1-\frac{(l+1)\Delta}{u^2})} : X_{1,x,u}(t_1) > 0\right\}}\\
&\sim \frac{\int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\sqrt{\Delta}]} : W_1(s) - \frac{1-a\rho}{1-\rho^2}s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x} dx \int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - \frac{1-a\rho}{1-\rho^2}s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x} dx \int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx}\\
&= \frac{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx}\\
&= \frac{\sqrt{\Delta}}{\Delta} \frac{\int_{\mathbb{R}} \frac{1}{\sqrt{\Delta}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \frac{1}{\Delta} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx}\\
&= \frac{\sqrt{\Delta}}{\Delta} > 0.
\end{aligned} \tag{12.3}$$

Hence combination of (12.1), (12.2) and (12.3) for large enough u leads to

$$\frac{S_{1,l}+S_{2,l}+S_{3,l}}{S_{0,l}} \le \frac{C}{\sqrt{\Delta}} + e^{-C\sqrt{\Delta}} + \frac{e^{-C_2\Delta}}{e^{C_2\Delta}-1},$$

which further implies

$$\lim_{\Delta\to\infty} \lim_{u\to\infty} \frac{S_{1,l}+S_{2,l}+S_{3,l}}{S_{0,l}} = 0$$

establishing the proof. □

13. Proof of negligibility of (3.17)

We consider the case $\bar{s}_u \le t_u$ and the proof of negligibility of R_3, since other cases follow from symmetry. For any $L_{1,u} \le l \le N_u - L_{1,u}$

$$\begin{aligned}
&\sum_{k=1}^{N_u-l} \mathbb{P}\left\{ \exists_{s_1\in E^1_{u,1}, s_2\in E^1_{u,k+l}, t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s_1)>u \\ W_1^*(s_2)>u \\ W_2^*(t_2)>u \end{array} \right\} \\
&= \frac{1}{u}\sum_{k=1}^{N_u-l} \int_{\mathbb{R}} \phi(u+c_1-\frac{x}{u}) \\
&\quad \times \mathbb{P}\left\{ \exists_{s_1\in E^1_{u,1}, s_2\in E^1_{u,k+l}, t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s_1)>u \\ W_1^*(s_2)>u \\ W_2^*(t_2)>u \end{array} \middle| W_1(1)=u+c_1-\frac{x}{u} \right\} dx \\
&= \frac{1}{u}\sum_{k=1}^{N_u-l} \int_{\mathbb{R}} \phi(u+c_1-\frac{x}{u}) \mathbb{P}\left\{ \exists_{s\in E^1_{u,1}} : W_1(s)-W_1(1)+c_1-c_1 s > \frac{x}{u} \right\} \\
&\quad \times \mathbb{P}\left\{ \exists_{s_2\in E^1_{u,k+l}, t_2\in E^2_{u,l}} : \begin{array}{c} W_1^*(s_2)>u \\ W_2^*(t_2)>u \end{array} \middle| W_1(1)=u+c_1-\frac{x}{u} \right\} dx \\
&= \frac{1}{u}\sum_{k=1}^{N_u-l} \int_{\mathbb{R}} \phi(u+c_1-\frac{x}{u}) \mathbb{P}\left\{ \exists_{s\in E^1_{u,1}} : W_1(s)-W_1(1)+c_1-c_1 s > \frac{x}{u} \right\} \\
&\quad \times \mathbb{P}\left\{ \exists_{s_2\in E^1_{u,k+l}, t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0 \right\} dx,
\end{aligned}$$

where $X_{x,u}(s_2,t_2) = (X_{1,x,u}(s_2), X_{2,x,u}(t_2))$ is a bivariate Gaussian process with

$$\mathbb{E}\{X_{x,u}(s_2,t_2)\} = -\begin{pmatrix} -c_1 s_2 + s_2(c_1-\frac{x}{u}) \\ -c_2 t_2 + \rho t_2 (c_1 - \frac{x}{u}) \end{pmatrix} + \begin{pmatrix} -(a-s_2)u \\ -(a-\rho t_2)u \end{pmatrix}$$

and

$$\Sigma_{X_{x,u}(s_2,t_2)} = \begin{pmatrix} s_2 - s_2^2 & s_2 - \rho s_2 t_2 \\ s_2 - \rho s_2 t_2 & t_2 - \rho^2 t_2^2 \end{pmatrix}.$$

Denote

$$S_{0,l} = \mathbb{P}\left\{ \exists_{t_2\in E^2_{u,l}} : X_{2,x,u}(t_2) > 0 \right\}, S_{1,l} = \sum_{k=2}^{N_u-l} \mathbb{P}\left\{ \exists_{s_2\in E^1_{u,k+l}, t_2\in E^2_{u,l}} : X_{x,u}(t_1,t_2)>0 \right\},$$

$$S_{2,l} = \mathbb{P}\left\{ \exists_{s_2\in(\bar{s}_u - \frac{(l+2)\Delta}{u^2}, \bar{s}_u - \frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}), t_2\in E^2_{u,l}} : X_{x,u}(s_2,t_2)>0 \right\},$$

$$S_{3,l} = \mathbb{P}\left\{ \exists_{s_2\in(\bar{s}_u - \frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}, \bar{s}_u - \frac{(l+1)\Delta}{u^2}), t_2\in E^2_{u,l}} : X_{x,u}(s_2,t_2)>0 \right\}.$$

Observe that for (3.17) to be negligible it is enough to show that for all $L_{1,u} \le l \le N_u$, as $u \to \infty, \Delta \to \infty$

$$\frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} \to 0.$$

Notice that for $X_{x,u}(s_2, t_2) = (X_{1,x,u}(s_2), X_{2,x,u}(t_2))$ we have

$$\begin{aligned}
\mathbb{P}\Big\{\exists_{s_2 \in E^1_{u,k+l}, t_2 \in E^2_{u,l}} : X_{x,u}(s_2, t_2) > 0\Big\} &\le \mathbb{P}\Big\{\exists_{s_2 \in E^1_{u,k+l}, t_2 \in E^2_{u,l}} : X_{1,x,u}(s_2) + X_{2,x,u}(t_2) > 0\Big\} \\
&\le \mathbb{P}\Big\{\exists_{s_2 \in E^1_{u,k+l}, t_2 \in E^2_{u,l}} : \frac{X_{1,x,u}(s_2) + X_{2,x,u}(t_2)}{\sigma_{k,u}} > 0\Big\},
\end{aligned}$$

where

$$\sigma^2_{k,u} := \max_{s_2 \in E^1_{u,k+l}, t_2 \in E^2_{u,l}} \sigma^2_u(s_2, t_2)$$

and $\sigma^2_u(s_2, t_2) := Var(X_{1,x,u}(s_2) + X_{2,x,u}(t_2))$. Then for any $s_2 \in E^1_{u,k+l}, t_2 \in E^2_{u,l}$

$$\frac{\partial \sigma^2_u(s_2, t_2)}{\partial s_2} = 3 - (2s_2 + 2t_2\rho) \sim 1 - 2\rho = 2, \frac{\partial \sigma^2_u(s_2, t_2)}{\partial t_2} = 1 - (2t_2\rho^2 + 2s_2\rho) \sim 1 - 2\rho^2 - 2\rho = \frac{3}{2}$$

as $u \to \infty$. Hence, since both derivatives are positive, $\sigma^2_{k,u}(t_1, t_2)$ attains its maximum at $s^*_2 = \bar{s}_u - \frac{(l+k)\Delta}{u^2}, t^*_2 = t_u - \frac{l\Delta}{u^2}$. Consequently

$$\sigma^2_{k,u} = 3\bar{s}_u + t_u - \bar{s}^2_u - \rho^2 t^2_u - 2\rho \bar{s}_u t_u - \frac{1}{u^2}(\frac{5l\Delta}{2} + 2k\Delta) + O(\frac{\Delta^2(k+l)^2}{u^4}) := \eta^2 - \frac{1}{u^2}(\frac{5l\Delta}{2} + 2k\Delta) + O(\frac{\Delta^2(k+l)^2}{u^4}).$$

Denote $\mu_u := \mathbb{E}\{X_{1,x,u}(s^*_2) + X_{2,x,u}(t^*_2)\} = 2u + c_1 s^*_2 + c_2 t^*_2 - (s^*_2 + \rho t^*_2)(u + c_1 - \frac{x}{u})$. Using [11][Thm 8.1], there exist constants $C, C_2 > 0$ such that

$$\begin{aligned}
S_{1,l} &\le \sum_{k=2}^{N_u - l} \mathbb{P}\Big\{\exists_{t_1 \in E^2_{u,k+l}, t_2 \in E^2_{u,l}} : \frac{X_{1,x,u}(t_1) + X_{2,x,u}(t_2)}{\sigma_{k,u}} > 0\Big\} \\
&\le \sum_{k=2}^{N_u - l} C \frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu^2_u}{2\sigma^2_{k,u}}} \\
&= \sum_{k=2}^{N_u - l} C \frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu^2_u(\eta^2 + \frac{1}{u^2}(\frac{5l\Delta}{2} + 2k\Delta) + O(\frac{1}{u^4}))}{2((4-4\rho^2)^2 + O(\frac{1}{u^4}))}} \\
&\le C \frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu^2_u(\eta^2 + \frac{1}{u^2}\frac{5l\Delta}{2} + O(\frac{1}{u^4}))}{2((\eta^2)^2 + O(\frac{1}{u^4}))}} \sum_{k=2}^{N_u - l} e^{-C_2 k(\Delta + O(\frac{1}{u^2}))} \\
&\le C \frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu^2_u(\eta^2 + \frac{1}{u^2}\frac{5l\Delta}{2} + O(\frac{1}{u^4}))}{2((\eta^2)^2 + O(\frac{1}{u^4}))}} \frac{e^{-C_2\Delta}}{e^{C_2\Delta} - 1}. \qquad (13.1)
\end{aligned}$$

Similarly we get that

$$S_{2,l} \le C \frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu^2_u(\eta^2 + \frac{1}{u^2}\frac{5l\Delta}{2} + O(\frac{1}{u^4}))}{2((\eta^2)^2 + O(\frac{1}{u^4}))}} e^{-C_2\sqrt{\Delta}}. \qquad (13.2)$$

Using Lemma 3.4 from [16] and Lemma 3.7 from [16] we have for any $l = O(\frac{u \log u}{\Delta})$ as $u \to \infty, \Delta \to \infty$

$$\begin{aligned}
\frac{S_{3,l}}{S_{0,l}} &\leq \frac{\mathbb{P}\left\{\exists_{t_2 \in (t_u - \frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}, t_u - \frac{(l+1)\Delta}{u^2})} : X_{2,x,u}(t_2) > 0\right\}}{\mathbb{P}\left\{\exists_{t_w \in (t_u - \frac{(l+2)\Delta}{u^2}, t_u - \frac{(l+1)\Delta}{u^2})} : X_{2,x,u}(t_2) > 0\right\}} \\
&\sim \frac{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \mathbb{P}\left\{\exists_{s\in[0,\Delta]} : W_1(s) - \frac{1-a\rho}{1-\rho^2} s > x\right\} e^{\frac{1-a\rho}{1-\rho^2}x} dx \int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx} \\
&= \frac{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx} \\
&= \frac{\sqrt{\Delta}}{\Delta} \frac{\int_{\mathbb{R}} \frac{1}{\sqrt{\Delta}} \mathbb{P}\{\exists_{t\in[0,\sqrt{\Delta}]} : W_2(t) - at > x\} e^{2ax} dx}{\int_{\mathbb{R}} \frac{1}{\Delta} \mathbb{P}\{\exists_{t\in[0,\Delta]} : W_2(t) - at > x\} e^{2ax} dx} \\
&= \frac{\sqrt{\Delta}}{\Delta} > 0. \qquad (13.3)
\end{aligned}$$

Hence combination of (13.1), (13.2) and (13.3) for large enough u leads to

$$\frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} \leq \frac{C}{\sqrt{\Delta}} + e^{-C\sqrt{\Delta}} + \frac{e^{-C_2\Delta}}{e^{C_2\Delta} - 1}$$

and further

$$\lim_{\Delta\to\infty} \lim_{u\to\infty} \frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} = 0$$

establishing the proof. □

14. Proof of case (v) of Theorem 2.2

Recall that with $F_u = [1 - \frac{\Delta \log u}{u}, 1] \times [t_u - \frac{\Delta \log u}{u}, t_u + \frac{\Delta \log u}{u}]$

$$\begin{aligned}
\pi_\rho(c_1, c_2; u, au) &\leq \sum_{k=1}^{N_u} \sum_{l=-N_u}^{N_u} \mathbb{P}\left\{\exists_{s\in E^1_{u,k}, t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\} \\
&+ \mathbb{P}\left\{\exists (s,t) \in [0,1]^2 \setminus F_u : \ W_1^*(s) > u, W_2^*(t) > au\right\} \\
&= \sum_{k=1}^{N_u} \sum_{l=-N_u}^{N_u} \mathbb{P}\left\{\exists_{s\in E^1_{u,k}, t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}(1 + o(1)).
\end{aligned}$$

Using Taylor expansion we have

$$u^2(q_{\boldsymbol{a}_u(k_u,l_u)}(k_u, l_u) - q_{\boldsymbol{a}_u(1,t_u)}(1, t_u)) = \tau_1(k-1)\Delta + \tau_4 \frac{(l-1)^2\Delta^2}{u^2} + o(\frac{k^2}{u^2}) + o(\frac{l^3}{u^4}),$$

where $\tau_1 = (1 - 2a\rho)^2 > 0, \tau_4 = -\frac{\rho^3(1-2a\rho)^4}{a(1-a\rho)} > 0$. Using Lemma 3.4 and the symmetry of the sum, we get as $u \to \infty$

$$\sum_{k=1}^{N_u} \sum_{l=-N_u}^{N_u} \mathbb{P}\left\{\exists_{s\in E^1_{u,k}, t\in E^2_{u,l}} : W_1^*(s) > u, W_2^*(t) > au\right\}$$

$$\sim \quad Iu^{-2}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1,au+c_2t^*)\sum_{k=1}^{N_u}\sum_{l=-N_u}^{N_u}e^{-\tau_1(k-1)\Delta}e^{-\frac{\tau_4}{2}\frac{l^2\Delta^2}{u^2}},$$

where

$$I=\int_{\mathbb{R}}\mathbb{P}\{\sup_{s\in[0,\Delta]}(W_1(s)-\frac{1-a\rho}{1-\rho^2t^*}s)>x\}e^{\frac{1-a\rho}{1-\rho^2t^*}x}dx\int_{\mathbb{R}}\mathbb{P}\{\sup_{t\in[0,\Delta]}(W_2(t)-\frac{a}{t^*}t)>x\}e^{2\frac{a}{t^*}x}dx.$$

Using Lemma 3.4 from [16] with Lemma 3.7 from [16], we get as $u\to\infty$

$$\begin{aligned}
&\sum_{k=1}^{N_u}\sum_{l=-N_u}^{N_u}\mathbb{P}\Big\{\exists_{s\in E^1_{u,k},t\in E^2_{u,l}}:W_1^*(s)>u,W_2^*(t)>au\Big\}\\
&\sim\quad 2u^{-1}\frac{1}{\sqrt{\tau_4}}\frac{1}{(1-e^{-\tau_1\Delta})}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1,au+c_2t^*)\frac{1-\rho^2t^*}{1-a\rho}\\
&\quad\times\int_{\mathbb{R}}\frac{1}{\Delta}\mathbb{P}\{\sup_{t\in[0,\Delta]}(W_2(t)-\frac{a}{t^*}t)>x\}e^{2\frac{a}{t^*}x}dx\sum_{l=-N_u}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{l^2\Delta^2}{u^2}}\\
&\sim\quad 2u^{-1}\frac{1}{\sqrt{\tau_4}}\frac{1-\rho^2t^*}{1-a\rho}\frac{a}{t^*}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1,au+c_2t^*)\sum_{l=-N_u}^{N_u}\frac{\sqrt{\tau_4}\Delta}{u}e^{-\frac{\tau_4}{2}\frac{l^2\Delta^2}{u^2}}\\
&\sim\quad \frac{2a\sqrt{2\pi}}{t^*\sqrt{\tau_4}}u^{-1}\frac{1-\rho^2t^*}{1-a\rho}e^{-a\frac{c_1^2\rho^2-2c_1c_2\rho+c_2^2}{2\rho(1-a\rho)}}\varphi_{t^*}(u+c_1,au+c_2t^*),\ \text{as }\Delta\to\infty.
\end{aligned}$$

To complete the proof, (3.18) needs to be shown to be asymptotically negligible, which is given below.

15. Proof of negligibility of (3.18)

For any $-N_u\le l\le N_u$

$$\begin{aligned}
&\sum_{k=1}^{N_u-l}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}:\begin{array}{c}W_1^*(s)>u\\ W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\right\}\\
&=\quad\frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\\
&\quad\times\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}:\begin{array}{c}W_1^*(s)>u\\ W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\middle|W_1(1)=u+c_1-\frac{x}{u}\right\}dx\\
&=\quad\frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\mathbb{P}\Big\{\exists_{s\in E^1_{u,1}}:W_1(s)-W_1(1)+c_1-c_1s>\frac{x}{u}\Big\}\\
&\quad\times\mathbb{P}\left\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}\begin{array}{c}W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\middle|W_1(1)=u+c_1-\frac{x}{u}\right\}dx
\end{aligned}$$

$$= \frac{1}{u}\sum_{k=1}^{N_u-l}\int_{\mathbb{R}}\phi(u+c_1-\frac{x}{u})\mathbb{P}\Big\{\exists_{s\in E^1_{u,1}}: W_1(s)-W_1(1)+c_1-c_1s>\frac{x}{u}\Big\}$$
$$\times\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\Big\}dx,$$

where $X_{x,u}(t_1,t_2)=(X_{1,x,u}(t_1),X_{2,x,u}(t_2))$ is a bivariate Gaussian process with

$$\mathbb{E}\{X_{x,u}(t_1,t_2)\} = -\begin{pmatrix} -c_2t_1+\rho t_1(c_1-\frac{x}{u}) \\ -c_2t_2+\rho t_2(c_1-\frac{x}{u})\end{pmatrix}+\begin{pmatrix} -(a-\rho t_1)u \\ -(a-\rho t_2)u\end{pmatrix}$$

and

$$\Sigma_{X_{x,u}(t_1,t_2)} = \begin{pmatrix} t_1-\rho^2t_1^2 & t_1-\rho^2t_1t_2 \\ t_1-\rho^2t_1t_2 & t_2-\rho^2t_2^2\end{pmatrix}.$$

Denote

$$S_{0,l}=\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,l},t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\Big\}, S_{1,l}=\sum_{k=2}^{N_u-l}\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\Big\},$$

$$S_{2,l}=\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2},1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}),t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\right\},$$
$$S_{3,l}=\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2},1-\frac{(l+1)\Delta}{u^2}),t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\right\}.$$

Observe that for (3.18) to be negligible it is enough to show that for all $-N_u\le l\le N_u$, as $u\to\infty,\Delta\to\infty$

$$\frac{S_{1,l}+S_{2,l}+S_{3,l}}{S_{0,l}}\to 0.$$

Notice that for $X_{x,u}(t_1,t_2)=(X_{1,x,u}(t_1),X_{2,x,u}(t_2))$ we have

$$\begin{aligned}\mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}: X_{x,u}(t_1,t_2)>0\Big\} &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}: X_{1,x,u}(t_1)+X_{2,x,u}(t_2)>0\Big\}\\ &\le \mathbb{P}\Big\{\exists_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}: \frac{X_{1,x,u}(t_1)+X_{2,x,u}(t_2)}{\sigma_{k,u}}>0\Big\},\end{aligned}$$

where

$$\sigma^2_{k,u}:=\max_{t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}}\sigma^2_u(t_1,t_2)$$

and $\sigma^2_u(t_1,t_2):=Var(X_{1,x,u}(t_1)+X_{2,x,u}(t_2))$. Then for any $t_1\in E^2_{u,k+l},t_2\in E^2_{u,l}$

$$\frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_1}=3-(2t_1\rho^2+2t_2\rho^2), \frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_2}=1-(2t_2\rho^2+2t_1\rho^2)$$

as $u\to\infty$.

(1) If $\frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_1}>0,\frac{\partial\sigma^2_u(t_1,t_2)}{\partial t_2}>0$, then $t_1^*=\frac{a}{\rho(2a\rho-1)}-\frac{(l+k)\Delta}{u^2}, t_2^*=\frac{a}{\rho(2a\rho-1)}-\frac{l\Delta}{u^2}$. Consequently

$$\sigma^2_{k,u} = -\frac{1}{\rho(2a\rho-1)^2}\left(4a-4a^2\rho^2-\frac{1}{u^2}(f_k+g_l)+O(\frac{\Delta^2(k+l)^2}{u^4})\right), \tag{15.1}$$

where $g_l=8al\Delta\rho^2-4l\Delta\rho$.

(2) If $\frac{\partial\ \sigma_u^2(t_1,t_2)}{\partial t_1} > 0, \frac{\partial \sigma_u^2(t_1,t_2)}{\partial t_2} < 0$, then $t_1^* = \frac{a}{\rho(2a\rho-1)} - \frac{(l+k)\Delta}{u^2}, t_2^* = \frac{a}{\rho(2a\rho-1)} - \frac{(l+1)\Delta}{u^2}$. Consequently (15.1) holds with $g_l = 8al\Delta\rho^2 - 4l\Delta\rho - \Delta\rho + 4a^2\Delta\rho^3$.

(3) If $\frac{\partial \sigma_u^2(t_1,t_2)}{\partial t_1} < 0, \frac{\partial \sigma_u^2(t_1,t_2)}{\partial t_2} < 0$, then $t_1^* = \frac{a}{\rho(2a\rho-1)} - \frac{(l+k+1)\Delta}{u^2}, t_2^* = \frac{a}{\rho(2a\rho-1)} - \frac{l\Delta}{u^2}$. Consequently (15.1) holds with $g_l = 8al\Delta\rho^2 - 4l\Delta\rho - 4\Delta\rho + 8a\Delta\rho^2$.

In all of the above scenarios $f_k = 8ak\Delta\rho^2 - 3k\Delta\rho - 4a^2k\Delta\rho^3$. Notice that for $1 > \frac{a}{\rho(2a\rho-1)} > 0$ we have $f_k > 0$ and $\rho < 0$. Denote $\mu_u := \mathbb{E}\{X_1(t_1^*) + X_2(t_2^*)\} = 2au + c_2t_1^* + c_2t_2^* - \rho(t_1^* + t_2^*)(u + c_1 - \frac{x}{u})$. For all $i \in 1, 2, 3$, using [11][Thm 8.1], there exist constants $C, C_2 > 0$ such that

$$\begin{aligned} S_{1,l} &\le \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu_u^2}{2\sigma_{k,u}^2}} \\ &= \sum_{k=2}^{N_u-l} C\frac{\mu_u}{\sigma_{k,u}} e^{-\frac{\mu_u^2\left(4a-4a^2\rho^2+\frac{1}{u^2}(f_k+g_l^{(i)})+O(\frac{1}{u^4})\right)}{-\frac{2}{\rho(2a\rho-1)^2}((4a-4a^2\rho^2)^2+O(\frac{1}{u^4}))}} \\ &\le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{g_l^{(i)}}{u^2}+O(\frac{1}{u^4}))}{-\frac{2}{\rho(2a\rho-1)^2}((4a-4a^2\rho^2)^2+O(\frac{1}{u^4}))}} \sum_{k=2}^{N_u-l} e^{-C_2k(\Delta+O(\frac{1}{u^2}))} \\ &\le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{g_l^{(i)}}{u^2}+O(\frac{1}{u^4}))}{-\frac{2}{\rho(2a\rho-1)^2}((4a-4a^2\rho^2)^2+O(\frac{1}{u^4}))}} \frac{e^{-C_2\Delta}}{e^{C_2\Delta}-1}. \end{aligned} \tag{15.2}$$

Similarly we get that

$$S_{2,l} \le C\frac{\mu_u}{\sigma_{0,u}} e^{-\frac{\mu_u^2(4-4\rho^2+\frac{g_l^{(i)}}{u^2}+O(\frac{1}{u^4}))}{-\frac{1}{\rho(2a\rho-1)^2}((4a-4a^2\rho^2)^2+O(\frac{1}{u^4}))}} e^{-C_2\sqrt{\Delta}} \tag{15.3}$$

and

$$\begin{aligned} S_{3,l} &\le \mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+1+\frac{1}{\sqrt{\Delta}})\Delta}{u^2}, 1-\frac{(l+1)\Delta}{u^2})} : X_1(t_1) > 0\right\} \\ &\le \frac{\sqrt{\Delta}}{\Delta}\mathbb{P}\left\{\exists_{t_1\in(1-\frac{(l+2)\Delta}{u^2}, 1-\frac{(l+1)\Delta}{u^2})} : X_1(t_1) > 0\right\}. \end{aligned} \tag{15.4}$$

Using (15.2), (15.3) and (15.4) we have that for some $C > 0$ and large enough u

$$\frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} \le \frac{C}{\sqrt{\Delta}} + e^{-C\sqrt{\Delta}} + \frac{e^{-C\Delta}}{e^{C\Delta}-1}$$

and hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty} \frac{S_{1,l} + S_{2,l} + S_{3,l}}{S_{0,l}} = 0.$$

Therefore, the proof follows. ∎

16. Proof of negligibility of (3.19)

For large enough u by taking $\boldsymbol{b}_u = \Sigma_{1,t_u}^{-1}(1+\frac{c_1}{u}, 1+\frac{c_2 t_u}{u})$, $\boldsymbol{b} = \Sigma_{1,t_u}^{-1}(1,1) > (0,0)$, the asymptotics given in the proof of case (v) of Theorem 2.2 imply that

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\Big\{\exists_{(s,t)\in F_{i,u}} : W_1^*(s) > u, W_2^*(t) > u\Big\} = -\frac{1}{2V_1},$$

for $i = 1,2$, where $V_1 := Var\left(\frac{b_1 W_1(1) + b_2 W_2(t^*)}{b_{1,u}(1+\frac{c_1}{u}) + b_{2,u}(1+\frac{c_2 t_u}{u})}\right) \sim Var\left(\frac{b_2 W_1(t^*) + b_1 W_2(1)}{b_{1,u}(1+\frac{c_1 t_u}{u}) + b_{2,u}(1+\frac{c_2}{u})}\right)$.

Moreover

$$\mathbb{P}\{\exists_{(s,t)\in F_{1,u},(s',t')\in F_{2,u}} : W_1^*(s) > u, W_2^*(t) > u, W_1^*(s') > u, W_2^*(t') > u\}$$
$$\le \mathbb{P}\left\{\exists_{(s,t)\in F_{1,u},(s',t')\in F_{2,u}} : \frac{b_1 W_1^*(s) + b_2 W_2^*(t)}{2(b_{1,u}(1+\frac{c_1}{u}) + b_{2,u}(1+\frac{c_2 t_u}{u}))} + \frac{b_2 W_1^*(s') + b_1 W_2^*(t')}{2(b_{1,u}(1+\frac{c_1 t_u}{u}) + b_{2,u}(1+\frac{c_2}{u}))} > u\right\}. \tag{16.1}$$

Since

$$\forall_{(s,t)\in F_{1,u}} \lim_{u\to\infty}(s,t) = (1,t^*), \forall_{(s,t)\in F_{2,u}} \quad \lim_{u\to\infty}(s,t) = (t^*,1)$$

and variance function of process under supremum in (16.1) is continuous, then using Borell-TIS inequality (see e.g., [11]) we get

$$\lim_{u\to\infty} \frac{\log \mathbb{P}\left\{\exists_{(s,t)\in F_{1,u},(s',t')\in F_{2,u}} : \frac{b_1 W_1^*(s) + b_2 W_2^*(t)}{2(b_{1,u}(1+\frac{c_1}{u}) + b_{2,u}(1+\frac{c_2 t_u}{u}))} + \frac{b_2 W_1^*(s') + b_1 W_2^*(t')}{2(b_{1,u}(1+\frac{c_1 t_u}{u}) + b_{2,u}(1+\frac{c_2}{u}))} > u\right\}}{u^2} \le -\frac{1}{2V_2},$$

where

$$V_2 := Var\left(\frac{b_1 W_1^*(1) + b_2 W_2^*(t^*)}{2(b_{1,u}(1+\frac{c_1}{u}) + b_{2,u}(1+\frac{c_2 t_u}{u}))} + \frac{b_2 W_1^*(t^*) + b_1 W_2^*(1)}{2(b_{1,u}(1+\frac{c_1 t_u}{u}) + b_{2,u}(1+\frac{c_2}{u}))}\right).$$

Since $t^* < 1$, for large enough u

$$\begin{aligned} V_1 - V_2 &\sim \frac{b_1^2(1-\rho) + b_2^2 t^*(1-\rho) - 2b_1 b_2 t^*(1-\rho)}{2(b_{1,u}+b_{2,u})^2} \\ &> (1-\rho)\frac{b_1^2 + b_2^2 t^* - 2b_1 b_2\sqrt{t^*}}{2(b_{1,u}+b_{2,u})^2} \\ &= (1-\rho)\frac{(b_1 - b_2\sqrt{t^*})^2}{2(b_{1,u}+b_{2,u})^2} \ge 0 \end{aligned}$$

Hence (3.19) is asymptotically negligible as $u \to \infty$.

17. Proof of negligibility of (A.1)

According to Lemma 3.1 t_u is the point that maximizes the function q. With the proof of Lemma 3.1 we have that $\sigma_u^2 - Var(Z_u(s,t))$ will be smallest for $(s,t) \in \partial F_u$. Additionally we can expand the function q to get that

$$\sigma_u^2 - Var(Z_u(s,t)) = \tau_1(1-s) + \tau_2(t_u - t) + \tau_3(1-s)^2 + \tau_4(t_u-t)^2 + \tau_5(1-s)(t_u-t) + o(1),$$

with $\tau_1, \tau_2 \geq 0$ and $\tau_3, \tau_4, \tau_5 > 0$, since t_u is the point that maximizes the function q. Since $(1-s), (t_u - t) = O(\frac{\log(u)}{u})$, then at worst

$$\sigma_u^2 - Var(Z_u(s,t)) > \tau \frac{\log(u)^2}{u^2},$$

which completes the proof. □

Acknowledgments

We are thankful to the reviewers for valuable comments and corrections which improved the presentation significantly. Partial support by SNSF Grant 200021-196888 is kindly acknowledged. K.D. was partially supported by NCN Grant No 2018/31/B/ST1/00370.

References

[1] G. A. Delsing, M. R. H. Mandjes, P. J. C. Spreij, and E. M. M. Winands, "Asymptotics and approximations of ruin probabilities for multivariate risk processes in a Markovian environment," *arXiv preprint arXiv:1812.09069*, 2018.

[2] K. Dębicki, E. Hashorva, and Z. Michna, "Simultaneous ruin probability for two-dimensional Brownian risk model," *J. Appl. Probab.*, vol. 57, no. 2, pp. 597–612, 2020.

[3] K. Dębicki and M. Mandjes, *Queues and Lévy Fluctuation Theory*. Springer International Publishing, 2015.

[4] S. Kou and H. Zhong, "First-passage times of two-dimensional Brownian motion," *Adv. Appl. Prob.*, vol. 48, pp. 1045–1060, 2016.

[5] H. He, W. Keirstead, and J. Rebholz, "Double lookbacks," *Mathematical Finance, Vol. 8, No. 3 (July 1998), 201–228*, 1998.

[6] A. Metzler, "On the first passage problem for correlated Brownian motion," *Statistics and Probability Letters*, vol. 80, pp. 277–284, 2010.

[7] L. C. G. Rogers and L. Shepp, "The correlation of the maxima of correlated Brownian motions," *J. Appl. Prob.*, vol. 43, no. 2, pp. 880–883, 2006.

[8] K. Dębicki, L. Ji, and T. Rolski, "Logarithmic asymptotics for probability of component-wise ruin in a two-dimensional Brownian model," *Risks*, vol. 7, no. 3, 2019.

[9] K. Dębicki, L. Ji, and T. Rolski, "Exact asymptotics of component-wise extrema of two-dimensional Brownian motion," *Extremes*, vol. 23, no. 4, pp. 569–602, 2020.

[10] D. Korshunov and L. Wang, "Tail asymptotics for Shepp-statistics of Brownian motion in $\mathbb{R}^d$," *Extremes*, vol. 23, no. 1, pp. 35–54, 2020.

[11] V. I. Piterbarg, *Asymptotic methods in the theory of Gaussian processes and fields*, vol. 148 of *Translations of Mathematical Monographs.* Providence, RI: American Mathematical Society, 1996.

[12] K. Dębicki, K. Kosiński, M. Mandjes, and T. Rolski, "Extremes of multidimensional Gaussian processes," *Stochastic Processes and their Applications*, vol. 120, no. 12, pp. 2289 – 2301, 2010.

[13] K. Dębicki, E. Hashorva, L. Ji, and T. Rolski, "Extremal behavior of hitting a cone by correlated Brownian motion with drift," *Stochastic Processes and their Applications*, vol. 128, no. 12, pp. 4171 – 4206, 2018.

[14] A. B. Dieker, "Extremes of Gaussian processes over an infinite horizon," *Stochastic Processes and their Applications*, vol. 115, no. 2, pp. 207 – 248, 2005.

[15] E. Hashorva, "Approximation of some multivariate risk measures for Gaussian risks," *Journal of Multivariate Analysis*, vol. 169, pp. 330–340, 2019.

[16] K. Dębicki, E. Hashorva, and K. Krystecki, "Finite-time ruin probability for correlated Brownian motions," *Submitted*, 2020.

Krzysztof Dębicki, Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Email address: Krzysztof.Debicki@math.uni.wroc.pl

Enkelejd Hashorva, Department of Actuarial Science, University of Lausanne, Chamberonne 1015, Lausanne, Switzerland

Email address: Enkelejd.Hashorva@unil.ch

Konrad Krystecki, Department of Actuarial Science, University of Lausanne, Chamberonne 1015, Lausanne, Switzerland and Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Email address: Konrad.Krystecki@unil.ch