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CUMULATIVE PARISIAN RUIN PROBABILITY FOR TWO-DIMENSIONAL BROWNIAN RISK MODEL

BY

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Abstract. Let $(W_1(s), W_2(t))$, $s, t \geqslant 0$, be a bivariate Brownian motion with standard Brownian motion marginals and constant correlation $\rho \in (-1, 1)$. We derive the exact asymptotics as $u \to \infty$ for the cumulative Parisian ruin probability

$$\mathbb{P}\left\{\begin{array}{l}\int_{[0,1]} \mathbf{1}(W_1(s) - c_1 s > u)\, ds > H_1(u) \\ \int_{[0,1]} \mathbf{1}(W_2(t) - c_2 t > au)\, dt > H_2(u)\end{array}\right\}$$

for $c_1, c_2 \in \mathbb{R}$, $a \in (0, 1]$ and suitably adjusted $H_1(u), H_2(u)$.

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1. INTRODUCTION

Consider the following Brownian risk model for two portfolios:

$$R_i(t) = u_i + c_i t - W_i(t), \quad i = 1, 2,$$

where the claims $W_i(t), t \geqslant 0$, are modeled by two dependent standard Brownian motions, initial capitals $u_i > 0$ and premium rates c_i. One-dimensional results for such risk models have been covered e.g. in [12, 6]. We model dependence between coordinates, as e.g. in [14, 15], through

$$(W_1(s), W_2(t)) = (B_1(s), \rho B_1(t) + \sqrt{1-\rho^2}\, B_2(t)), \quad s, t \geqslant 0,$$

where B_1, B_2 are two independent standard Brownian motions and $\rho \in (-1, 1)$. The probability of ruin of a single portfolio in a finite time horizon was given e.g. in [12]. Denote $W_i^*(s) = W_i(s) - c_i s$, $B_i^*(s) = B_i(s) - c_i s$, $i = 1, 2$. In the literature several models describing the ruin have been introduced and investigated for the multidimensional setting. For example, define the simultaneous ruin probability as

$$\overline{\pi}_{A,\rho}(c_1, c_2, u, au) = \mathbb{P}\{\exists_{s\in A}\, W_1^*(s) > u,\, W_2^*(s) > au\}$$

which has recently been studied in [7] for $A = [0,1]$ and in [4] in higher dimensions. Similarly, define the non-simultaneous ruin probability as

$$\pi_{A\times B,\rho}(c_1, c_2, u, au) = \mathbb{P}\left\{\exists_{s\in A,\, t\in B}\ \begin{matrix} W_1^*(s) > u \\ W_2^*(t) > au \end{matrix}\right\},$$

which has been studied for $A = B = [0,1]$ in [5] and for $A = B = [0,\infty)$ in [9]. The setting of a single crossing has been extended to different functionals to reflect less strict approaches to the definition of ruin. One of the most popular extensions of the notion of ruin is the so-called Parisian ruin, which for a two-dimensional non-simultaneous model is defined as

$$\begin{aligned}&\mathcal{P}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au)\\ &\qquad := \mathbb{P}\left\{\exists_{(s,t)\in A\times B}\ \forall_{s'\in[s,s+H_1(u)],\, t'\in[t,t+H_2(u)]}\ \begin{matrix} W_1^*(s') > u \\ W_2^*(t') > au \end{matrix}\right\}\end{aligned}$$

for some $\boldsymbol{H}(u) = (H_1(u), H_2(u)) \geqslant \mathbf{0}$ and compact sets A, B. Notice that without loss of generality we can assume $a \leqslant 1$. This model has been widely investigated in various settings; see e.g. [20, 18]. In [13] this approach has been further modified to allow for the time above the threshold to come from disjoint intervals. This leads to the definition of the cumulative Parisian ruin

$$\mathcal{S}_{A,H(u)}(c, u) := \mathbb{P}\left\{\int_A \mathbf{1}(W_1^*(s) > u)\, ds > H(u)\right\},$$

$$\mathcal{S}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au) := \mathbb{P}\left\{\begin{matrix} \int_A \mathbf{1}(W_1^*(s) > u)\, ds > H_1(u) \\ \int_B \mathbf{1}(W_2^*(t) > au)\, dt > H_2(u) \end{matrix}\right\},$$

for some $\boldsymbol{H}(u) = (H_1(u), H_2(u)) \geqslant \mathbf{0}$ and compact sets A, B. The cumulative Parisian ruin is also known under the name of sojourn model and finds various applications in finance and insurance, e.g. in [21, 16, 19], and has been widely investigated in the context of Gaussian processes (see e.g. [17, 11]). In the one-dimensional context it was proven that for the choice of $H(u) = S/u^2$, $S > 0$, the cumulative Parisian ruin differs from the classical ruin only by a multiplicative constant. In [18] we have seen that for a non-simultaneous model, the Parisian ruin follows the same pattern. In [2, 17] it is proven that for different setups the Parisian and cumulative Parisian ruin differ by a constant. We aim to prove it for the non-simultaneous ruin as well, for the arbitrary choice of $\boldsymbol{H}(u) = (S_1/u^2, S_2/u^2)$, which is closely connected to the behaviour of variance for the Brownian motion (see [22]). Notice that for $\boldsymbol{H}(u) = o(1/u^2)$, following the same line of proof, we find that the cumulative Parisian ruin is asymptotically equivalent to the classical ruin, which is a natural result, since the required period of crossing the barrier is so short that it is enough that the process crosses it once to stay there for the required time. On the other hand, if we choose $\boldsymbol{H}(u)$ such that $u^2 H_i(u) \to \infty$ for $u \to \infty$ with $H_i(u) < 1$, $i = 1, 2$, then we cannot employ the usual methods of finding the exact asymptotics. Whenever $\lim_{u\to\infty} H_i(u) > 0$ for some $i \in \{1, 2\}$, if $\rho < 0$ then the processes behave vastly differently and require a different formulation of the optimization problem.

1.1. Notation and preliminaries. Let $X \overset{d}{=} Y$ denote equality in distribution of the random variables X and Y. Further for functions $f(x), g(x)$ we write

$$f(x) \sim g(x) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Let

$$\Sigma_{s,t} = \begin{pmatrix} s & \rho \min(s,t) \\ \rho \min(s,t) & t \end{pmatrix}$$

be the covariance matrix of $(W_1(s), W_2(t))$. We denote

$$\boldsymbol{a} = (1, a)^\top, \quad q_{\boldsymbol{a}}(s,t) := \boldsymbol{a}^\top \Sigma_{s,t}^{-1} \boldsymbol{a}, \quad \boldsymbol{b}(s,t) := \Sigma_{s,t}^{-1} \boldsymbol{a}$$

and set

$$q^*_{\boldsymbol{a}}(s,t) = \min_{\boldsymbol{x} \geqslant \boldsymbol{a}} q_{\boldsymbol{x}}(s,t), \quad q^*_{\boldsymbol{a}} = \min_{s,t\in[0,1]} q^*_{\boldsymbol{a}}(s,t). \tag{1.1}$$

This quadratic optimization problem plays an important role in e.g. [10, 1, 3, 8]. It is well known that

$$\lim_{u\to\infty} \frac{1}{u^2} \log \mathbb{P}\{\exists_{s,t\in[0,1]}\, W_1^*(s) > u,\, W_2^*(t) > au\} = -\frac{q^*_{\boldsymbol{a}}}{2}. \tag{1.2}$$

Hence the function q plays a crucial role in calculating the asymptotics and our aim is to find its optimal point. However, it was found to be insufficient to determine the exact asymptotics in the non-simultaneous setting. In [5] instead of $\boldsymbol{a}$, $\boldsymbol{a}_u = (1 + c_1 s/u, a + c_2 t/u)^\top$ was used and it was crucial in understanding the exact asymptotics. Therefore in the second part of this contribution we use

$$q_{\boldsymbol{a}_u}(s,t) := \boldsymbol{a}_u^\top \Sigma_{s,t}^{-1} \boldsymbol{a}_u, \quad \boldsymbol{b}_u(s,t) := \Sigma_{s,t}^{-1} \boldsymbol{a}_u$$

and the corresponding optimization problem. Notice that for logarithmic asymptotics, using $\boldsymbol{a}$ is no different from using $\boldsymbol{a}_u$, since by using Taylor expansion for the function q, we find

$$\lim_{u\to\infty} \frac{\log(q_{\boldsymbol{a}_u})}{\log(q_{\boldsymbol{a}})} = 1.$$

However, for the exact asymptotics we need the complete $\boldsymbol{a}_u$, as seen in Lemmas 3.1 and 3.2.

Notice that for any compact sets A, B the cumulative Parisian ruin generalizes the classical ruin since

$$\pi_{A\times B,\rho}(c_1, c_2, u, au) = \mathcal{S}_{A\times B,\mathbf{0}}(c_1, c_2, u, au).$$

Additionally, there is the following relation between the classical ruin and both types of Parisian ruin:

$$\begin{aligned}\pi_{A\times B,\rho}(c_1, c_2, u, au) &\geqslant \mathcal{S}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au) \\ &\geqslant \mathcal{P}_{A\times B,\boldsymbol{H}(u)}(c_1, c_2, u, au),\end{aligned} \tag{1.3}$$

which allows for simple proofs of finiteness and positivity of the constants arising in this note.

In order to understand the relation between the classical and cumulative Parisian ruin we often write

$$\mathscr{S}_{[0,1]^2,\boldsymbol{H}(u)}(c_1, c_2, u, au) := \mathbb{P}\left\{\begin{matrix}\int\limits_{[0,1]} \mathbf{1}(W_1^*(s) > u)\,ds > H_1(u) \\ \int\limits_{[0,1]} \mathbf{1}(W_2^*(t) > au)\,dt > H_2(u)\end{matrix}\;\middle|\; \exists_{v,w\in[0,1]}\ \begin{matrix} W_1^*(v) > u \\ W_2^*(w) > au\end{matrix}\right\},$$

which is an equivalent form to presenting the results for $\mathcal{S}_{S_1,S_2}(c_1, c_2, u, au)$, since

$$\mathscr{S}_{S_1,S_2}(c_1, c_2, u, au) = \frac{\mathcal{S}_{S_1,S_2}(c_1, c_2, u, au)}{\pi_{[0,1]^2,\rho}(c_1, c_2, u, au)},$$

where to further simplify the notation we write

$$\begin{aligned}\mathscr{S}_{S_1,S_2}(c_1, c_2, u, au) &:= \mathscr{S}_{[0,1]^2,(S_1/u^2,S_2/u^2)}(c_1, c_2, u, au), \\ \mathcal{S}_{S_1,S_2}(c_1, c_2, u, au) &:= \mathcal{S}_{[0,1]^2,(S_1/u^2,S_2/u^2)}(c_1, c_2, u, au).\end{aligned}$$

2. MAIN RESULTS

We begin with dimension-reduction cases, where one of the coordinates dominates the other and the results are up to a constant the same as in one dimension. This behavior was already observed in [5] for the classical ruin and in [18] for the Parisian ruin.

THEOREM 2.1. *If $a \leqslant \rho$, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2, u, au) = \frac{(2 + S_1)\Phi(\sqrt{S_1/2}) - \sqrt{S_1/\pi}\, e^{-S_1/4}}{2}.$$

Before proceeding to the analysis of the case $a > \rho$ we introduce constants that appear in the asymptotics below:

$$\widehat{\mathcal{P}}(w_1, w_2, S) := \int_{\mathbb{R}} \mathbb{P}\Big\{ \int_{[0,\infty)} \mathbf{1}(B_1(s) - w_1 s > x)\, ds > S \Big\} e^{w_2 x}\, dx,$$

$$\widehat{\mathcal{H}}(w_1, w_2, S) := \lim_{\Delta\to\infty} \int_{\mathbb{R}} \frac{1}{\Delta} \mathbb{P}\Big\{ \int_{[0,\Delta]} \mathbf{1}(B_1(t) - w_1 t > x)\, dt > S \Big\} e^{w_2 x}\, dx,$$

$$\widehat{\mathcal{R}}_{S_1,S_2} := \int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{l} \int_{[0,\infty)} \mathbf{1}(W_1(s) - s > x)\, ds > S_1 \\ \int_{[0,\infty)} \mathbf{1}(W_2(t) - at > y)\, dt > S_2 \end{array} \right\} e^{\frac{1-a\rho}{1-\rho^2}x + \frac{a-\rho}{1-\rho^2}y}\, dx\, dy.$$

In each particular case, finiteness and positivity of $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{H}}$ have been proven in Lemma 3.3. Let $A_a = \frac{1}{4a}(1 - \sqrt{8a^2 + 1})$. It represents the line which splits the cases of the main theorem. If $\rho > A_a$, then we observe a behaviour of coordinates working together to cross their respective barriers, since the correlation is either positive or slightly negative. In the opposite case, the coordinates compete and we observe quite a different behaviour.

THEOREM 2.2. *Let $\rho \in (-1, 1)$ and $a \in (\max(0, \rho), 1]$ be given.*

(i) *If $\rho > A_a$, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2; u, au) = \frac{\widehat{\mathcal{R}}_{S_1,S_2}}{\widehat{\mathcal{R}}_{0,0}}. \tag{2.1}$$

(ii) *If $\rho = A_a$ and $a < 1$, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2; u, au) = \frac{(1 - a\rho)\widehat{\mathcal{P}}\big(\frac{1-a\rho}{1-\rho^2}, \frac{1-a\rho}{1-\rho^2}, S_1\big)\widehat{\mathcal{H}}(a, 2a, S_2)}{2a(1 - \rho^2)}. \tag{2.2}$$

(iii) *If $\rho = A_a$ and $a = 1$, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2; u, au) = \frac{C_{3,1}C'_{3,1} + C_{3,2}C'_{3,2}}{C_3}, \tag{2.3}$$

where $C_{3,1} = \widehat{\mathcal{P}}(2, 2, S_1)\widehat{\mathcal{H}}(1, 2, S_2), C_{3,2} = \widehat{\mathcal{P}}(2, 2, S_2)\widehat{\mathcal{H}}(1, 2, S_1)$ and

$$C'_{3,1} = \begin{cases} e^{-2\frac{(\frac{1}{2}c_1 + c_2)^2}{3}} \Phi\big(c_2 + \frac{1}{2}c_1\big), & -\frac{1}{2}c_1 < c_2, \\ 1, & otherwise, \end{cases}$$

$$C'_{3,2} = \begin{cases} e^{-2\frac{(\frac{1}{2}c_2 + c_1)^2}{3}} \Phi\big(c_1 + \frac{1}{2}c_2\big), & -\frac{1}{2}c_2 < c_1, \\ 1, & otherwise, \end{cases}$$

$$C_3 = \begin{cases} e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\big(c_2+\frac{1}{2}c_1\big) \\ \quad + e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\big(c_1+\frac{1}{2}c_2\big), & c_2 > \max\big(-\frac{1}{2}c_1, -2c_1\big) \\ e^{-2\frac{(\frac{1}{2}c_1+c_2)^2}{3}}\Phi\big(c_2+\frac{1}{2}c_1\big) + \frac{1}{2}, & -\frac{1}{2}c_1 < c_2 \leqslant -2c_1, \\ \frac{1}{2} + e^{-2\frac{(\frac{1}{2}c_2+c_1)^2}{3}}\Phi\big(c_1+\frac{1}{2}c_2\big), & -2c_1 < c_2 \leqslant -\frac{1}{2}c_1, \\ 1, & c_2 \leqslant \min\big(-\frac{1}{2}c_1, -2c_1\big). \end{cases}$$

(iv) *If* $a < 1$ *and* $\rho < A_a$*, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2; u, au) = -\frac{\widehat{\mathcal{P}}\big(\frac{1-a\rho}{1-\rho^2 t_*}, \frac{1-a\rho}{1-\rho^2 t_*}, S_1\big)\widehat{\mathcal{H}}\big(\frac{a}{t_*}, \frac{2a}{t_*}, S_2\big)}{2\rho}. \tag{2.4}$$

(v) *If* $a = 1$ *and* $\rho < A_a$*, then*

$$\lim_{u\to\infty} \mathscr{S}_{S_1,S_2}(c_1, c_2; u, au) = -\frac{C_5}{2\rho}, \tag{2.5}$$

where

$$C_5 = \begin{cases} \widehat{\mathcal{P}}\big(\frac{1-\rho}{1-\rho^2 t_*}, \frac{1-\rho}{1-\rho^2 t_*}, S_1\big)\widehat{\mathcal{H}}\big(\frac{1}{t_*}, \frac{2}{t_*}, S_2\big), & c_1 \leqslant c_2, \\ \widehat{\mathcal{P}}\big(\frac{1-\rho}{1-\rho^2 t_*}, \frac{1-\rho}{1-\rho^2 t_*}, S_2\big)\widehat{\mathcal{H}}\big(\frac{1}{t_*}, \frac{2}{t_*}, S_1\big), & c_1 > c_2, \end{cases} \qquad t_* = \frac{1}{\rho(2\rho-1)}.$$

3. PROOFS

We begin with the proofs of dimension-reduction cases, where we prove that one of the coordinates only contributes to a constant. Since the asymptotics of $\pi_{[0,1]^2,\rho}(c_1, c_2, u, au)$ has already been studied in [5] it is often easier to analyze the asymptotics of $\mathcal{S}_{S_1,S_2}(c_1, c_2, u, au)$, and hence in the proofs we focus on the behaviour of the latter.

3.1. Proof of Theorem 2.1. CASE (i): $a < \rho$. Notice that

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1, c_2, u, au) \leqslant \mathcal{S}_{[0,1],S_1/u^2}(c_1, u).$$

Moreover, with $H(u,t) := \frac{(a-\rho)u+(c_2-\rho c_1)t}{\sqrt{1-\rho^2}}$ we have, for large enough u,

$$\mathcal{S}_{[0,1]^2,\boldsymbol{S}/u^2}(c_1, c_2, u, au)$$
$$\geqslant \mathbb{P}\left\{ \begin{array}{l} \int_{[0,1]} \mathbf{1}\begin{pmatrix} B_1^*(s) > u \\ \rho B_1(s) + \sqrt{1-\rho^2}\, B_2(s) - c_2 s > au \end{pmatrix} ds > S_1/u^2 \\ \int_{[0,1]} \mathbf{1}\begin{pmatrix} B_1^*(s) > u - 1/\sqrt{u} \\ \rho B_1(s) + \sqrt{1-\rho^2}\, B_2(s) - c_2 s > au \end{pmatrix} ds > \max(S_1, S_2)/u^2 \end{array} \right\}$$

$$
\geqslant \mathbb{P}\left\{\begin{array}{l}
\int_{[0,1]} \mathbf{1}\left(\begin{array}{l} B_1^*(s)>u \\ \rho(u+c_1 s)+\sqrt{1-\rho^2}\,B_2(s)-c_2 s>au\end{array}\right) ds > S_1/u^2 \\
\int_{[0,1]} \mathbf{1}\left(\begin{array}{l} B_1^*(s)>u-1/\sqrt{u} \\ \rho(u-1/\sqrt{u}+c_1 s)+\sqrt{1-\rho^2}\,B_2(s)-c_2 s>au\end{array}\right) ds \\
\qquad\qquad\qquad\qquad\qquad\qquad > \max(S_1,S_2)/u^2
\end{array}\right\}
$$

$$
\geqslant \mathbb{P}\left\{\begin{array}{l}
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u)\mathbf{1}(\forall_{t\in[0,1]}\, B_2(t)>H(u,t))\,ds > S_1/u^2 \\
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u-1/\sqrt{u}\,)\mathbf{1}(\forall_{t\in[0,1]}\, B_2(t)>H(u,t))\,ds \\
\qquad\qquad\qquad\qquad\qquad\qquad > \max(S_1,S_2)/u^2
\end{array}\right\}
$$

$$
= \mathbb{P}\left\{\begin{array}{l}
\mathbf{1}(\forall_{t\in[0,1]}\, B_2(t)>H(u,t)) \int_{[0,1]} \mathbf{1}(B_1^*(s)>u)\,ds > S_1/u^2 \\
\mathbf{1}(\forall_{t\in[0,1]}\, B_2(t)>H(u,t)) \int_{[0,1]} \mathbf{1}(B_1^*(s)>u-1/\sqrt{u}\,)\,ds \\
\qquad\qquad\qquad\qquad\qquad\qquad > \max(S_1,S_2)/u^2
\end{array}\right\}
$$

$$
= \mathbb{P}\big\{\forall_{t\in[0,1]}\, B_2(t)>H(u,t)\big\}
\times \mathbb{P}\left\{\begin{array}{l}
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u)\,ds > S_1/u^2 \\
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u-1/\sqrt{u}\,)\,ds > \max(S_1,S_2)/u^2
\end{array}\right\}.
$$

Since $a<\rho$, we have

$$
\lim_{u\to\infty} \mathbb{P}\{\forall_{t\in[0,1]}\, B_2(t)>H(u,t)\} = 1.
$$

Further by the independence of increments and self-similarity of Brownian motion we find that for $\widehat{B_1}$ a standard Brownian motion independent of B_1 and for

$$
t_{\inf} = \begin{cases} \inf\{s\in[0,1]: B_1^*(s)=u\} & \text{if } \exists_{t\in[0,1]}\, B_1^*(t)>u, \\ 1 & \text{otherwise,} \end{cases}
$$

we have

$$
\text{(3.1)}\qquad \mathbb{P}\left\{\begin{array}{l}
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u)\,ds > S_1/u^2 \\
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u-1/\sqrt{u})\,ds > \max(S_1,S_2)/u^2
\end{array}\right\}
$$

$$
= \mathbb{P}\left\{\begin{array}{l}
\int_{[t_{\inf},1]} \mathbf{1}(B_1^*(s)>u)\,ds > S_1/u^2 \\
\int_{[0,1]} \mathbf{1}(B_1^*(s)>u-1/\sqrt{u})\,ds > \max(S_1,S_2)/u^2
\end{array}\right\}
$$

$$
\geqslant \mathbb{P}\left\{\begin{array}{l}
\int_{[t_{\inf},1]} \mathbf{1}(B_1^*(s)-B_1^*(t_{\inf})>0)\,ds > S_1/u^2 \\
\forall_{s\in(t_{\inf}-\max(0,S_2-S_1)/u^2,\,t_{\inf})}\, B_1^*(t_{\inf})-B_1^*(s) < 1/\sqrt{u}
\end{array}\right\}
$$

$$= \mathbb{P}\Bigg\{ \int\limits_{[t_{\inf},1]} \mathbf{1}(B_1^*(s) - B_1^*(t_{\inf}) > 0)\, ds > S_1/u^2 \Bigg\}$$
$$\times\, \mathbb{P}\big\{\forall_{s\in(t_{\inf}-\max(0,S_2-S_1)/u^2,t_{\inf})}\ B_1^*(t_{\inf}) - B_1^*(s) < 1/\sqrt{u}\big\}$$
$$= \mathbb{P}\Bigg\{ \int\limits_{[t_{\inf},1]} \mathbf{1}(B_1^*(s) - B_1^*(t_{\inf}) + B_1^*(t_{\inf}) > u)\, ds > S_1/u^2 \Bigg\}$$
$$\times\, \mathbb{P}\{\forall_{s\in(0,\max((S_2-S_1)/u^2,0))}\ \widehat{B_1}(s) + c_1 s < 1/\sqrt{u}\}$$
$$= \mathbb{P}\Bigg\{ \int\limits_{[0,1]} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2 \Bigg\}$$
$$\times\, \mathbb{P}\{\forall_{s\in[0,\max(S_2-S_1,0)]}\ \widehat{B_1}(s) + c_1 s/u < \sqrt{u}\}.$$

Finally, we have

$$\lim_{u\to\infty} \mathbb{P}\big\{\forall_{s\in[0,\max(S_2-S_1,0)]}\ B_1(s) + c_1 s/u < \sqrt{u}\big\} = 1.$$

One-dimensional asymptotics found in [11, Cor. 3.2, Thm. 3.7] gives

$$\mathbb{P}\Big\{ \int\limits_{[0,1]} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2 \Big\} \sim \int\limits_{\mathbb{R}} e^x \mathbb{P}\Big\{ \int\limits_0^\infty \mathbf{1}(B(t) - t > x)\, dt > S_1 \Big\}\, dx\, \Psi(u + c_1) \sim ((2 + S_1)\Phi(\sqrt{S_1/2}) - \sqrt{S_1/\pi}\, e^{-S_1/4})\Psi(u + c_1), \tag{3.2}$$

and hence the proof of case (i) is complete.

CASE (ii): $a = \rho$. For $\overline{\Delta}(u) = [1 - 1/\sqrt{u}, 1]$, we have

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1, c_2; u, au) \leqslant \mathcal{S}_{\overline{\Delta}(u)^2,\boldsymbol{S}/u^2}(c_1, c_2; u, au) + \pi_{[0,1]^2\setminus\overline{\Delta}(u)^2,\rho}(c_1, c_2; u, au) =: \mathbb{S} + \pi_{[0,1]^2\setminus\overline{\Delta}(u)^2,\rho}(c_1, c_2; u, au).$$

Furthermore,

$$\mathbb{S} \leqslant \mathbb{P}\left\{ \begin{array}{l} \int\limits_{\overline{\Delta}(u)} \mathbf{1}(W_1^*(s) > u)\, ds > S_1/u^2 \\ \int\limits_{\overline{\Delta}(u)} \mathbf{1}(W_2^*(t) > au)\, dt > S_2/u^2 \\ \forall_{v\in\overline{\Delta}(u)}\ W_1^*(v) < u + 1/\sqrt{u} \end{array} \right\} + \mathbb{P}\big\{\exists_{v\in\overline{\Delta}(u)}\ W_1^*(v) > u + 1/\sqrt{u}\big\}$$
$$=: \mathbb{S}_1 + \mathbb{S}_2.$$

Further,

$$\mathbb{S}_1 \leqslant \mathbb{P}\left\{\begin{array}{l} \int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2 \\ \int_{\overline{\Delta}(u)} \mathbf{1}\left(B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t > \frac{-\rho}{\sqrt{u}\sqrt{1-\rho^2}}\right) dt > \frac{S_2}{u^2} \\ \forall_{v \in \overline{\Delta}(u)}\ B_1^*(v) < u + 1/\sqrt{u} \end{array}\right\}$$

$$\leqslant \mathbb{P}\Big\{\int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2\Big\}$$

$$\times\, \mathbb{P}\Big\{\int_{\overline{\Delta}(u)} \mathbf{1}\left(B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t > \frac{-\rho}{\sqrt{u}\sqrt{1-\rho^2}}\right) dt > \frac{S_2}{u^2}\Big\}$$

$$\leqslant \mathbb{P}\Big\{\int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2\Big\}$$

$$\times\, \mathbb{P}\left\{\exists_{t \in \overline{\Delta}(u)}\ B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t > \frac{-\rho}{\sqrt{u}\sqrt{1-\rho^2}}\right\}.$$

From [11, Thm. 3.7] we have

$$\mathbb{P}\Big\{\int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\, ds > S_1/u^2\Big\} \sim \mathcal{S}_{[0,1],S_1/u^2}(c_1, u).$$

Additionally, for large enough u,

$$\mathbb{P}\left\{\exists_{t \in \overline{\Delta}(u)}\ B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t > \frac{-\rho}{\sqrt{u}\sqrt{1-\rho^2}}\right\} \geqslant \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right).$$

On the other hand, for $h_u = 1 - \frac{1}{\sqrt{u}}$ and $\bar{H}(u) = \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} h_u - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} - \frac{1}{u^{1/8}}$,

$$\mathbb{P}\left\{\exists_{t \in \overline{\Delta}(u)}\ B_2(t) - \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t > \frac{-\rho}{\sqrt{u}\sqrt{1-\rho^2}}\right\}$$

$$= \mathbb{P}\left\{\exists_{t \in \overline{\Delta}(u)}\ B_2(t) > \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} \,\middle|\, B_2(h_u) > \bar{H}(u)\right\}$$

$$\times\, \mathbb{P}\left\{B_2(h_u) > \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} h_u - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} - \frac{1}{u^{1/8}}\right\}$$

$$+\, \mathbb{P}\left\{\exists_{t \in \overline{\Delta}(u)}\ B_2(t) > \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} t - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} \,\middle|\, B_2(h_u) < \bar{H}(u)\right\}$$

$$\times\, \mathbb{P}\left\{B_2(h_u) < \frac{c_2 - \rho c_1}{\sqrt{1-\rho^2}} h_u - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} - \frac{1}{u^{1/8}}\right\}.$$

Notice that

$$\lim_{u\to\infty} \mathbb{P}\left\{ B_2(h_u) > \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}} h_u - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} - \frac{1}{u^{1/8}} \right\} = \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right).$$

Further, by self-similarity and independence of increments of Brownian motion,

$$\begin{aligned}\mathbb{P}&\left\{ \exists_{t\in\overline{\Delta}(u)}\ B_2(t) > \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}} t - \frac{\rho}{\sqrt{u}\sqrt{1-\rho^2}} \;\middle|\; B_2(h_u) < \bar{H}(u) \right\}\\ &\leqslant \mathbb{P}\left\{ \exists_{t\in[0,1/\sqrt{u}]}\ B_2(t) > \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}} t + \frac{1}{u^{1/8}} \right\}\\ &= \mathbb{P}\left\{ \exists_{t\in[0,1]}\ B_2(t) > \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}} \frac{t}{u^{1/4}} + u^{1/8} \right\}.\end{aligned}$$

As $u\to\infty$, the above tends to 0 and therefore

$$\lim_{u\to\infty} \mathbb{P}\left\{ \exists_{t\in\overline{\Delta}(u)}\ B_2(t) - \frac{c_2-\rho c_1}{\sqrt{1-\rho^2}} t > 0 \right\} = \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right).$$

From [5, Thm. 2.1] we have

$$\mathbb{S}_2 = o(\pi_{\overline{\Delta}(u),\rho}(c_1,u)).$$

Further by [11, Thm. 3.7] we infer that for some $C>0$,

$$\lim_{u\to\infty} \frac{\mathcal{S}_{[0,1],S_1/u^2}(c_1,u)}{\pi_{[0,1],\rho}(c_1,u)} = C,$$

and hence as $u\to\infty$,

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1,c_2;u,au) \leqslant \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right) \mathcal{S}_{[0,1],S_1/u^2}(c_1,u).$$

For the lower bound notice that similarly to the case $a<\rho$,

$$\begin{aligned}\mathcal{S}_{[0,1]^2,S/u^2}(c_1,c_2;u,au) &\geqslant \mathbb{P}\left\{ \forall_{t\in\overline{\Delta}(u)}\ B_2(t) > \frac{(c_2-\rho c_1)t}{\sqrt{1-\rho^2}} \right\}\\ &\quad\times \mathbb{P}\left\{ \begin{array}{l} \int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\,ds > S_1/u^2 \\ \int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u - 1/\sqrt{u})\,ds > \max(S_1,S_2)/u^2 \end{array} \right\}.\end{aligned}$$

With (3.1) we have

$$\mathbb{P}\left\{ \begin{array}{l} \int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u)\,ds > S_1/u^2 \\ \int_{\overline{\Delta}(u)} \mathbf{1}(B_1^*(s) > u - 1/\sqrt{u})\,ds > \max(S_1,S_2)/u^2 \end{array} \right\} \sim \mathcal{S}_{[0,1],S_1/u^2}(c_1,u).$$

Moreover

$$\mathbb{P}\left\{\forall_{t\in\overline{\Delta}(u)}\ B_2(t) > \frac{(c_2-\rho c_1)t}{\sqrt{1-\rho^2}}\right\} \sim \Phi\left(\frac{\rho c_1 - c_2}{\sqrt{1-\rho^2}}\right).$$

Finally, by [5, Thm. 2.1] we have

$$\pi_{[0,1]^2\setminus\overline{\Delta}(u)^2,\rho}(c_1,c_2;u,au) = o(\pi_{[1-\Delta/u^2,1],\rho}(c_1,u)),$$

and therefore the asymptotics follows from (3.2). □

3.2. Proof of Theorem 2.2. To understand the behaviour of the ruin probability, we need to find the solution of the optimization problem for $q^*_{a_u}(s,t)$, which has been presented in [5, Lemma 3.1]:

LEMMA 3.1. *For all large u we have:*

(i) *If $a=1$ and $\rho\in(-1,-1/2)$, then $q^*_{a_u}(s,t)$ attains its local minima on $[0,1]^2$ only at*

$$(s_u,t_u) := \left(1, \frac{1}{\rho(2\rho-1)+\frac{c_2-\rho c_1}{u}}\right), \qquad (\bar{s}_u,\bar{t}_u) := \left(\frac{1}{\rho(2\rho-1)+\frac{c_1-\rho c_2}{u}}, 1\right).$$

(ii) *If $a=1$ and $\rho=-\frac{1}{2}$, then $q^*_{a_u}(s,t)$ attains its local minima on $[0,1]^2$ only at*

$$(s_u,t_u) := \left(1, \min\left(\frac{1}{1+\frac{c_2+2c_1}{u}}, 1\right)\right),$$
$$(\bar{s}_u,\bar{t}_u) := \left(\min\left(\frac{1}{1+\frac{c_1+2c_2}{u}}, 1\right), 1\right).$$

(iii) *For any other $a\in(\max(0,\rho),1]$ and $\rho\in(-1,1)$, $q^*_{a_u}(s,t)$ attains its unique minimum on $[0,1]^2$ at*

$$(s_u,t_u) := \begin{cases} \left(1, \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\right) & \textit{if } \frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}} \in [0,1], \\ (1,1) & \textit{otherwise}. \end{cases}$$

We further denote $t^* := \lim_{u\to\infty} t_u$. As in the proof of Theorem 2.1 we can again focus on the asymptotics of $\mathcal{S}_{S_1,S_2}(c_1,c_2,u,au)$. The following lemma tackles the behaviour of the probability of ruin on square areas of side length Δ/u^2 for $\Delta>0$. In the following we use

$$E_{u,k,l} = \left[1-\frac{k\Delta}{u^2}, 1-\frac{(k-1)\Delta}{u^2}\right] \times \left[t_u-\frac{l\Delta}{u^2}, t_u-\frac{(l-1)\Delta}{u^2}\right], \qquad E=[-\Delta,0],$$

$$k_u = 1 - \frac{(k-1)\Delta}{u^2}, \quad l_u = t_u - \frac{(l-1)\Delta}{u^2},$$

and we denote by $\varphi_{s,t}(x,y)$ the probability density function of $(W_1^*(s), W_2^*(t))$ at (x,y) and set $\varphi_{t_u}(x,y) := \varphi_{1,t_u}(x,y)$. Additionally let us introduce a shorter notation $\eta_{u,k,l}(s,t) := (\eta_{1,u,k}(s), \eta_{2,u,l}(t)) := u(W_1(s/u^2 + k_u) - W_1(k_u) - c_1 s/u^2, W_2(t/u^2 + l_u) - W_2(l_u) - c_2 t/u^2)$.

LEMMA 3.2. *Let $\rho \in (-1,1), a \in (\max(0,\rho), 1]$, $l, k = O(u\log(u)/\Delta)$ and let $\Delta, S_1, S_2 > 0$ be given constants. Set $\mu_u = u^{-2}\varphi_{t_u}(u + c_1, au + c_2 t_u)$. Then, as $u \to \infty$,*

$$\mathcal{S}_{E_{u,k,l},(S_1,S_2)/u^2}(c_1, c_2, u, au) \sim \mu_u I_2(\Delta) e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u}(k_u,l_u) - q_{\boldsymbol{a}_u}(1,t_u))},$$

where

$$I_2(\Delta) = \begin{cases} \int\limits_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{matrix} \int\limits_{[0,\Delta]} \mathbf{1}(W_1(s) - s > x)\, ds > S_1 \\ \int\limits_{[0,\Delta]} \mathbf{1}(W_2(t) - at > y)\, dt > S_2 \end{matrix} \right\} e^{\lambda_1 x + \lambda_2 y}\, dx\, dy, & l_u = k_u, \\ \int\limits_{\mathbb{R}^2} \mathbb{P}\Big\{ \int\limits_{[0,\Delta]} \mathbf{1}(W_1(s) - s > x)\, ds > S_1 \Big\} \\ \quad \times \mathbb{P}\Big\{ \int\limits_{[0,\Delta]} \mathbf{1}\Big(W_2(t) - \frac{a-\rho}{t^* - \rho^2} t > y\Big)\, dt > S_2 \Big\} e^{\lambda_1 x + \lambda_2 y}\, dx\, dy, \\ & l_u > k_u, \\ \int\limits_{\mathbb{R}^2} \mathbb{P}\Big\{ \int\limits_{[0,\Delta]} \mathbf{1}\Big(W_1(s) - \frac{1-a\rho}{1-\rho^2 t^*} s > x\Big)\, ds > S_1 \Big\} \\ \quad \times \mathbb{P}\Big\{ \int\limits_{[0,\Delta]} \mathbf{1}\Big(W_2(t) - \frac{a}{t^*} t > y\Big)\, dt > S_2 \Big\} e^{\lambda_1 x + \lambda_2 y}\, dx\, dy, \\ & l_u < k_u, \end{cases}$$

and

$$\lambda_1 = \begin{cases} \frac{1}{t^*} \frac{1-a\rho}{1-\rho^2}, & l_u = k_u, \\ \frac{t^* - a\rho}{t^* - \rho^2}, & l_u > k_u, \\ \frac{1-a\rho}{1-\rho^2 t^*}, & l_u < k_u, \end{cases} \qquad \lambda_2 = \begin{cases} \frac{1}{t^*} \frac{a-\rho}{1-\rho^2}, & l_u = k_u, \\ \frac{a-\rho}{t^* - \rho^2}, & l_u > k_u, \\ \frac{a - \rho t^*}{t^* - \rho^2 (t^*)^2}, & l_u < k_u. \end{cases}$$

Moreover,

(3.3)

$$\lim_{u\to\infty} \sup_{l,k=O(u\log u)} \int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{l} \int_E \mathbf{1}(\eta_{1,u,k}(s) > x)\, ds > S_1 \\ \int_E \mathbf{1}(\eta_{2,u,l}(t) > y)\, dt > S_2 \end{array} \middle| \begin{array}{l} W_1^*(k_u) = u - x/u \\ W_2^*(l_u) = au - y/u \end{array} \right\} \times e^{\lambda_1 x + \lambda_2 y}\, dx\, dy < \infty.$$

Proof. We have

$$\begin{aligned}
&\mathcal{S}_{E_{u,k,l},(S_1,S_2)/u^2}(c_1, c_2, u, au) \\
&= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{l} \int_E \mathbf{1}(W_1^*(s/u^2 + k_u) > u)\, ds > S_1 \\ \int_E \mathbf{1}(W_2^*(t/u^2 + l_u) > au)\, dt > S_2 \end{array} \middle| \begin{array}{l} W_1^*(k_u) = u - x/u \\ W_2^*(l_u) = au - y/u \end{array} \right\} \\
&\qquad \times u^{-2} \varphi_{k_u,l_u}(u + c_1 k_u - x/u, au + c_2 l_u - y/u)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \mathbb{P}\Bigg\{ \int_E \mathbf{1}(W_1^*(s/u^2 + k_u) - W_1(k_u) + W_1(k_u) > u)\, ds > S_1, \\
&\qquad \int_E \mathbf{1}(W_2^*(t/u^2 + l_u) - W_2(l_u) + W_2(l_u) > au)\, dt > S_2 \ \Bigg| \ \begin{array}{l} W_1^*(k_u) = u - x/u \\ W_2^*(l_u) = au - y/u \end{array} \Bigg\} \\
&\qquad \times u^{-2} \varphi_{k_u,l_u}(u + c_1 k_u - x/u, au + c_2 l_u - y/u)\, dx\, dy \\
&= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{l} \int_E \mathbf{1}(\eta_{1,u,k}(s) > x)\, ds > S_1 \\ \int_E \mathbf{1}(\eta_{2,u,l}(t) > y)\, dt > S_2 \end{array} \middle| \begin{array}{l} W_1^*(k_u) = u - x/u \\ W_2^*(l_u) = au - y/u \end{array} \right\} \\
&\qquad \times u^{-2} \varphi_{k_u,l_u}(u + c_1 k_u - x/u, au + c_2 l_u - y/u)\, dx\, dy.
\end{aligned}$$

Notice that [5, Lemma 3.3] gives, for both $k_u > l_u$ and $k_u < l_u$, as $u \to \infty$,

$$\varphi_{k_u,l_u}(u + c_1 k_u - x/u, au + c_2 l_u - y/u) \sim \varphi_{t_u}(u + c_1, au + c_2 t_u) \times e^{-\frac{1}{2}u^2(q_{\boldsymbol{a}_u}(k_u,l_u) - q_{\boldsymbol{a}_u}(1,t_u))} e^{\lambda_1 x + \lambda_2 y}.$$

Next we investigate the behaviour of the process

$$\left\{ \begin{array}{l} \eta_{1,u,k}(s) \\ \eta_{2,u,l}(t) \end{array} \middle| \begin{array}{l} W_1^*(k_u) = u - x/u \\ W_2^*(l_u) = au - y/u \end{array} \right\}.$$

This process has already been studied in [5, Lemma 3.3], where its distribution was determined to be Gaussian with parameters depending on whether $k_u < l_u$, $k_u = l_u$ or $k_u > l_u$. However, to use the aforementioned calculations we need to prove that (3.3) is finite, so that we can use the dominated convergence theorem.

This comes straightforwardly from the combination of (1.3) and the fact that [5, (3.8)] is finite. This completes the proof. ■

Before we turn to the main proof, we introduce two lemmas that tackle the finiteness of the constants that come from Lemma 3.2.

LEMMA 3.3. (i) *For any* $b, c > 0$ *and* $S \geqslant 0$ *such that* $2b > c$ *we have*

$$\int_{\mathbb{R}} \mathbb{P}\Big\{ \int_{[0,\infty)} \mathbf{1}(W_1(t) - bt > x)\, dt > S \Big\} e^{cx}\, dx \in (0, \infty).$$

(ii) *For any* $b > 0$ *and* $S \geqslant 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{\mathbb{R}} \mathbb{P}\Big\{ \int_{[0,T]} \mathbf{1}(W_1(t) - bt > x)\, dt > S \Big\} e^{2bx}\, dx \in (0, \infty).$$

Finiteness and positivity of the one-dimensional constants have been proven in [11, proof of Thm. 3.4].

LEMMA 3.4. *Take any* $a > \max(0, \rho)$ *and* $S_1, S_2 \geqslant 0$. *Then*

$$\int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{l} \int_{\mathbb{R}_+} \mathbf{1}(W_1(s) - s > x)\, ds > S_1 \\ \int_{\mathbb{R}_+} \mathbf{1}(W_2(t) - at > y)\, dt > S_2 \end{array} \right\} e^{\lambda_1 x + \lambda_2 y}\, dx\, dy \in (0, \infty)$$

where $\lambda_1 = \frac{1-a\rho}{1-\rho^2}$, $\lambda_2 = \frac{a-\rho}{1-\rho^2}$.

Finiteness and positivity of constants above come straightforwardly from (1.3) and the constants from [5, Lemma 3.5] and [18, Thm. 2.2].

Proof of Theorem 2.2. Introduce

$$N_u := \left\lfloor \frac{u \log(u)}{\Delta} \right\rfloor, \qquad E^1_{u,m} := [(m+1)_u, m_u], \qquad E^2_{u,j} := [(j+1)_u, j_u],$$
$$K_u^{(1)} = \frac{(c_2 - c_1\rho)u}{\Delta}, \qquad K_u^{(2)} = \frac{(c_1 - c_2\rho)u}{\Delta},$$

where $m_u = 1 - (m-1)\Delta/u^2$, $j_u = t_u - (j-1)\Delta/u^2$. Different cases in the theorem are the result of various types of behaviour of the function $q^*_{a_u}(s,t)$ around the optimizing point.

CASE (i): $\rho > \frac{1}{4a}(1 - \sqrt{8a^2+1})$. According to [5, Lemma 3.1], $t^* = 1$. Denote $F_u := E^2_{u,1}$. For $\Delta > 0$, we have

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1, c_2; u, au) \geqslant \mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1, c_2; u, au).$$

On the other hand,

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1,c_2;u,au) \leqslant \mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1,c_2;u,au) + \pi_{[0,1]^2\setminus F_u}(c_1,c_2;u,au).$$

Using Lemmas 3.2 and 3.4 and taking $u \to \infty$ and $\Delta \to \infty$, we get

$$\mathcal{S}_{E^2_{u,1},(S_1,S_2)/u^2}(c_1,c_2;u,au) \sim \int_{\mathbb{R}^2} \mathbb{P}\left\{ \begin{array}{c} \int_{[0,\infty)} \mathbf{1}(W_1(s)-s>x)\,ds > S_1 \\ \int_{[0,\infty)} \mathbf{1}(W_2(t)-at>y)\,dt > S_2 \end{array} \right\} \times e^{\lambda_1 x+\lambda_2 y}\,dx\,dy\,u^{-2}\varphi_1(u+c_1,au+c_2).$$

From [5, Thm. 2.2] we have

$$\begin{aligned}\pi_{[0,1]^2\setminus E^2_{u,1}}(c_1,c_2;u,au) &= o(u^{-2}\varphi_1(u+c_1,au+c_2)) \\ &= o(\mathcal{S}_{E^2_{u,1},(S_1,S_2)/u^2}(c_1,c_2;u,au)).\end{aligned}$$

Thus, the proof of case (i) is complete.

CASE (ii): $\rho = \frac{1}{4a}(1-\sqrt{8a^2+1})$. From [18, Thm. 2.2, case (ii)] recall that

$$u^2(q_a(k_u,l_u)-q_a(1,1)) = \tau_1(k-1)\Delta + \tau_4\frac{(l-1)^2\Delta^2}{u^2} + o\left(\frac{k^2}{u^2}\right) + o\left(\frac{l^3}{u^4}\right),$$

where $\tau_1 = \frac{(1-a\rho)^2}{(1-\rho^2)^2} > 0$ and $\tau_4 = \frac{\rho^2-2a\rho^3+a^2\rho^2}{(1-\rho^2)^2} > 0$. The constants τ_i will vary for the other cases, analogously to what was calculated in [18, Thm. 2.2]. This case is split into two subcases. First consider $c_2-\rho c_1 \leqslant 0$. According to [5, Lemma 3.1], $t^*=1$. For $F_u := [1-\Delta/u^2,1]\times[1-\log(u)/u, 1-\Delta/u^2]$ we write

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1,c_2;u,au) \geqslant \mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1,c_2;u,au).$$

Since either the whole period of crossing the barrier occurs on F_u or we cross the barrier at least once on $[0,1]^2\setminus F_u$, it follows that

$$\mathcal{S}_{[0,1]^2,(S_1,S_2)/u^2}(c_1,c_2;u,au) \leqslant \mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1,c_2;u,au) + \pi_{[0,1]^2\setminus F_u}(c_1,c_2;u,au).$$

Using the Bonferroni inequality we obtain

$$\begin{aligned}
(3.4)\qquad & \mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1,c_2;u,au) \\
&\geqslant \sum_{l=2}^{N_u}\mathbb{P}\Big\{\int_{E^1_{u,1}}\mathbf{1}(W_1^*(s)>u)\,ds>\frac{S_1}{u^2},\int_{E^2_{u,l}}\mathbf{1}(W_2^*(t)>au)\,dt>\frac{S_2}{u^2}\Big\} \\
&\quad-\sum_{l=2}^{N_u}\sum_{m=l+1}^{N_u}\mathbb{P}\left\{\exists_{s\in E^1_{u,1},t_1\in E^2_{u,l},t_2\in E^2_{u,m}}\begin{array}{r} W_1^*(s)>u\\ W_2^*(t_1)>au\\ W_2^*(t_2)>au\end{array}\right\} \\
&=: S_{u,\Delta}-D_{u,\Delta},
\end{aligned}$$

Further, we have

$$\mathcal{S}_{F_u,(S_1,S_2)/u^2}(c_1,c_2;u,au)\leqslant S_{u,\Delta}+D_{u,\Delta}. \tag{3.5}$$

From Lemma 3.2 we get, as $u\to\infty$,

$$S_{u,\Delta}\sim C^{(1)}_{2,\mathcal{S}}(\Delta)C^{(2)}_{2,\mathcal{S}}(\Delta)u^{-2}\varphi_{t^*}(u+c_1,au+c_2)\sum_{l=2}^{N_u}e^{-\frac{1}{2}u^2(q_{a_u}(k_u,l_u)-q_{a_u}(1,1))},$$

where

$$\begin{aligned}
C^{(1)}_{2,\mathcal{S}}(\Delta)&=\int_{\mathbb{R}}\mathbb{P}\Big\{\int_{[0,\Delta]}\mathbf{1}\Big(W_1(s)-\frac{1-a\rho}{1-\rho^2}s>x\Big)\,ds>S_1\Big\}e^{\frac{1-a\rho}{1-\rho^2}x}\,dx,\\
C^{(2)}_{2,\mathcal{S}}(\Delta)&=\int_{\mathbb{R}}\mathbb{P}\Big\{\int_{[0,\Delta]}\mathbf{1}(W_2(t)-at>x)\,dt>S_2\Big\}e^{2ax}\,dx.
\end{aligned}$$

Using Taylor expansions together with [5, Lemma 3.6] we get, as $u\to\infty$,

$$\begin{aligned}
S_{u,\Delta}&\sim C^{(1)}_{2,\mathcal{S}}(\Delta)C^{(2)}_{2,\mathcal{S}}(\Delta)u^{-2}\varphi_{t^*}(u+c_1,au+c_2)\sum_{l=2}^{N_u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&=\frac{1}{\sqrt{\tau_4}}C^{(1)}_{2,\mathcal{S}}(\Delta)\frac{C^{(2)}_{2,\mathcal{S}}(\Delta)}{\Delta}u^{-1}\varphi_{t^*}(u+c_1,au+c_2)\sum_{l=2}^{N_u}\frac{\sqrt{\tau_4}\,\Delta}{u}e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}\\
&\sim C^{(1)}_{2,\mathcal{S}}(\Delta)\frac{C^{(2)}_{2,\mathcal{S}}(\Delta)}{\Delta}\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi_{t^*}(u+c_1,au+c_2).
\end{aligned}$$

With Lemma 3.3 we have

$$\lim_{\Delta\to\infty}C^{(1)}_{2,\mathcal{S}}(\Delta)=C^{(1)}_{2,\mathcal{S}}(\infty)\in(0,\infty),\qquad \lim_{\Delta\to\infty}\frac{C^{(2)}_{2,\mathcal{S}}(\Delta)}{\Delta}=C^{(2)}_{2,\mathcal{S}}(\infty)\in(0,\infty).$$

Hence

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{S_{u,\Delta}}{C^{(1)}_{2,\mathcal{S}}(\infty)C^{(2)}_{2,\mathcal{S}}(\infty)\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi_{t^*}(u+c_1,au+c_2)}=1.$$

From [5, Theorem 2.2, case (ii)] we have

$$(3.6)\quad \lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{D_{u,\Delta}}{S_{u,\Delta}} = \lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{D_{u,\Delta}}{C^{(1)}_{2,\mathcal{S}}(\infty)C^{(2)}_{2,\mathcal{S}}(\infty)\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi_{t^*}(u+c_1,au+c_2)} = 0$$

and also

$$(3.7)\quad \lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\pi_{[0,1]^2\setminus F_u}(c_1,c_2;u,au)}{S_{u,\Delta}} = \lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{\pi_{[0,1]^2\setminus F_u}(c_1,c_2;u,au)}{C^{(1)}_{2,\mathcal{S}}(\infty)C^{(2)}_{2,\mathcal{S}}(\infty)\frac{\sqrt{\pi}}{\sqrt{2\tau_4}}u^{-1}\varphi_{t^*}(u+c_1,au+c_2)} = 0.$$

Therefore, using the asymptotics for $\pi_{[0,1]^2}(c_1,c_2;u,au)$ from [5, Thm. 2.2] we obtain

$$\lim_{u\to\infty}\mathscr{S}_{S_1,S_2}(c_1,c_2;u,au) = \frac{(1-a\rho)\widehat{\mathcal{P}}\big(\frac{1-a\rho}{1-\rho^2},\frac{1-a\rho}{1-\rho^2},S_1\big)\widehat{\mathcal{H}}(a,2a,S_2)}{2a(1-\rho^2)}.$$

It remains to consider the case $c_2-\rho c_1>0$. Using [5, Lemma 3.1] we find that the only minimizer of $q^*_{a_u}(s,t)$ on $[0,1]^2$ is $(s_u,t_u)=\big(1,\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\big)$ with

$$\frac{a}{\rho(2a\rho-1)+\frac{c_2-\rho c_1}{u}}\nearrow 1$$

as $u\to\infty$. The optimal area according to [5, Thm. 2.2, case (iii)] is $F_u := [1-\Delta/u^2,1]\times[t_u-\log(u)/u,1-\Delta/u^2]$. From Lemma 3.2 we have, as $u\to\infty$,

$$S_{u,\Delta} \sim C^{(1)}_{2,\mathcal{S}}(\Delta)C^{(2)}_{2,\mathcal{S}}(\Delta)u^{-2}\varphi_{t^*}(u+c_1,au+c_2)\sum_{l=-K_u^{(1)}}^{N_u} e^{-\frac{1}{2}u^2(q_{a_u}(k_u,l_u)-q_{a_u}(1,1))},$$

where $C^{(1)}_{2,\mathcal{S}}(\Delta), C^{(2)}_{2,\mathcal{S}}(\Delta)$ are defined as above. Using Taylor expansions and [5, Lemma 3.6] we deduce that, as $u\to\infty$,

$$S_{u,\Delta}\sim C^{(1)}_{3,\mathcal{S}}(\Delta)C^{(2)}_{3,\mathcal{S}}(\Delta)u^{-2}\varphi_{t^*}(u+c_1,au+c_2)\sum_{l=-K_u^{(1)}}^{N_u} e^{-\frac{\tau_4}{2}\frac{(l-1)^2\Delta^2}{u^2}}$$
$$\sim C^{(1)}_{2,\mathcal{S}}(\Delta)\frac{C^{(2)}_{2,\mathcal{S}}(\Delta)}{\Delta}\frac{\sqrt{2\pi}}{\sqrt{\tau_4}}e^{-a\frac{(c_1\rho-c_2)^2}{2\rho(1-a\rho)}}\Phi(c_2-\rho c_1)u^{-1}\varphi_{t^*}(u+c_1,au+c_2).$$

Using (3.5) and (3.4) together with [5, Thm. 2.2, case (iii)] we deduce that

$$\lim_{\Delta\to\infty}\lim_{u\to\infty}\frac{S_{u,\Delta}}{C^{(1)}_{2,\mathcal{S}}C^{(2)}_{2,\mathcal{S}}\frac{\sqrt{2\pi}}{\sqrt{\tau_4}}e^{-a\frac{(c_1\rho-c_2)^2}{2\rho(1-a\rho)}}\Phi(c_2-\rho c_1)u^{-1}\varphi_{t^*}(u+c_1,au+c_2)}=1$$

and that (3.6) and (3.7) hold. Together with the asymptotics for $\pi_{[0,1]^2}(c_1,c_2;u,au)$ from [5, Thm. 2.2] we have

$$\lim_{u\to\infty}\mathscr{S}_{S_1,S_2}(c_1,c_2;u,au)=\frac{(1-a\rho)\widehat{\mathcal{P}}\big(\frac{1-a\rho}{1-\rho^2},\frac{1-a\rho}{1-\rho^2},S_1\big)\widehat{\mathcal{H}}(a,2a,S_2)}{2a(1-\rho^2)},$$

which completes the proof of case (ii).

The other cases follow the same path of proof, where the varying component is the main area F_u that we use in (3.5) and (3.4). Hence the proofs for the remaining cases are omitted. This completes the proof. □

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