ON FIRST ORDER AMENABILITY

EHUD HRUSHOVSKI, KRZYSZTOF KRUPIŃSKI, AND ANAND PILLAY

Abstract. We introduce the notion of *first order amenability*, as a property of a first order theory T: every complete type over $\emptyset$, in possibly infinitely many variables, extends to an automorphism-invariant global Keisler measure in the same variables. Amenability of T follows from amenability of the (topological) group $\mathrm{Aut}(M)$ for all sufficiently large $\aleph_0$-homogeneous countable models M of T (assuming T to be countable), but is radically less restrictive.

First, we study basic properties of amenable theories, giving many equivalent conditions. Then, applying a version of the stabilizer theorem from [13], we prove that if T is amenable, then T is G-compact, namely Lascar strong types and Kim-Pillay strong types over $\emptyset$ coincide. This extends and essentially generalizes a similar result proved via different methods for ω-categorical theories in [24]. In the special case when amenability is witnessed by $\emptyset$-definable global Keisler measures (which is for example the case for amenable ω-categorical theories), we also give a different proof, based on stability in continuous logic.

Parallel (but easier) results hold for the notion of *extreme amenability*.

0. Introduction

We introduce the notions of *amenable* and *extremely amenable* first order theory. This is part of our attempt to extract the model-theoretic content of the circle of ideas around [extreme] amenability of automorphism groups of countable structures, which we discuss further below. We say that T is *amenable* if for every $p \in S_{\bar{x}}(\emptyset)$, in any (possibly infinite) tuple of variables $\bar{x}$, there exists an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_p(\mathfrak{C}) := \{q \in S_{\bar{x}}(\mathfrak{C}) : p \subseteq q\}$, where $\mathfrak{C}$ is a monster model of T. *Extreme amenability* of T means that the invariant measure above can be chosen to be a *Dirac*, namely: every $p \in S_{\bar{x}}(\emptyset)$ extends to a global $\mathrm{Aut}(\mathfrak{C})$-invariant complete type. We study properties of [extreme] amenability, showing for example that they are indeed properties of the theory (i.e. do not depend on $\mathfrak{C}$) and providing several equivalent definitions. We will discuss here amenability, leaving the extreme version to further paragraphs. One of the equivalent definitions of amenability of T is that $\mathrm{Aut}(\mathfrak{C})$ is *relatively amenable* (i.e. there

2010 *Mathematics Subject Classification.* 03C45, 43A07.

Key words and phrases. Amenable theory, G-compactness.

The second author is supported by National Science Center, Poland, grants 2015/19/B/ST1/01151, 2016/22/E/ST1/00450, and 2018/31/B/ST1/00357.

The third author was supported by NSF grants DMS-136702, DMS-1665035, and DMS-1760212.

is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$ treated as a subset of $\mathfrak{C}^{\mathfrak{C}}$). Relative amenability of $\mathrm{Aut}(\mathfrak{C})$ (or, more generally, of the group of automorphisms of any model) is a natural counterpart of definable amenability of a definable group. The above observations work for any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$. For such an M, if $\mathrm{Aut}(M)$ is amenable as a topological group (with the pointwise convergence topology), then T is amenable. We point out in a similar fashion that (for countable T) if $\mathrm{Aut}(M)$ is amenable for all sufficiently large $\aleph_0$-homogeneous countable models, then T is amenable. In the NIP context, we get a full characterization of amenability of T in various terms, e.g. by saying that $\emptyset$ is an extension base, which also yields a class of examples of amenable theories, e.g. all stable or o-minimal or c-minimal theories are amenable. Also, the theories of measurable structures in the sense of Elwes and Macpherson (e.g. pseudo-finite fields) [7] are amenable.

This paper is concerned with the implications of [extreme] amenability of a first order theory T for the Galois group $\mathrm{Gal}_L(T)$. So let us discuss briefly those Galois groups as well as the notions of G-compactness and G-triviality and why they should be considered important. Formal definitions will be given in Section 1, but we give a rather more relaxed description now. See also the introduction to [24]. At the centre are the key notions of *strong types*. Two tuples $\bar{a}$ and $\bar{b}$ from the monster model $\mathfrak{C}$, of the same (bounded) length, have the same *Lascar strong type* if $E(\bar{a}, \bar{b})$ whenever E is an $\mathrm{Aut}(\mathfrak{C})$-invariant equivalence relation with boundedly many classes. If we instead consider only bounded equivalence relations E which are *type-definable* over $\emptyset$, we obtain the notion of having the same *Kim-Pillay strong type* (in short, KP-strong type). The group of permutations of all Lascar strong types induced by $\mathrm{Aut}(\mathfrak{C})$ is called the *Lascar Galois group* $\mathrm{Gal}_L(T)$; $\mathrm{Gal}_{KP}(T)$ is defined analogously. When Lascar strong types coincide with KP-strong types, $\mathrm{Gal}_L(T)$ has naturally the structure of a compact Hausdorff group, and T is said to be *G-compact*. When Lascar strong types coincide with types (over $\emptyset$), then $\mathrm{Gal}_L(T)$ is trivial, and T is said to be *G-trivial*. Lascar strong types present *obstructions* to various kinds of type amalgamation. Also in [26], where the Lascar Galois group was first defined, they present obstacles to recovering an ω-categorical theory T from its category of models. As KP-strong types are much easier to handle than Lascar strong types, G-compactness is a desirable property. In any case, $\mathrm{Gal}_L(T)$ and $\mathrm{Gal}_{KP}(T)$ are important invariants of an arbitrary complete first order theory T and play important roles in model theory.

The main result of this paper (proved in Section 4) is the following

Theorem 0.1. *Every amenable theory is G-compact.*

This result is a wide generalization of Theorem 0.7 from [24] which says that whenever M is a countable, ω-categorical structure and $\mathrm{Aut}(M)$ is amenable as a topological group, then $\mathrm{Th}(M)$ is G-compact. Theorem 0.7 of [24] was deduced (by a non-trivial argument which is interesting in its own right) from [24, Theorem 0.5], more precisely, from the fact that amenability of a topological group implies equality of certain model-theoretic/topological connected components. In [24], this last fact was proved for groups possessing a basis of open neighborhoods of the identity consisting of open subgroups, which was sufficient in the proof of [24, Theorem 0.7], because $\mathrm{Aut}(M)$ has this property; later, this fact was proved in full generality in [13, Corollary 2.37]. As to our very general Theorem 0.1, we do not have an argument showing that it follows from [13, Corollary 2.37]; instead we give a direct proof working with *relatively type-definable subsets* of the group of automorphisms of the monster model and using a version from [13] of Massicot-Wagner stabilizer theorem [28]. Theorem 0.1 can be viewed as a transposition of [28] from definable groups to theories. It might have followed easier from an application of [28] to the automorphism group of the monster model, if the "stabilizers" produced in [28] were quantifier-free definable. But they are not, and so we have to proceed differently.

In Section 3 (see Proposition 3.11), we give a completely different proof of Theorem 0.1 (with a better bound on the diameters of Lascar strong types than the one obtained in Theorem 4.1) which is based on stability theory in continuous logic, but under the stronger assumption of the existence of $\emptyset$-definable Keisler measures on all $\emptyset$-definable sets (in which case we say that the theory is *definably amenable*). This also includes the ω-categorical context from [24, Theorem 0.7], yielding yet another proof of [24, Theorem 0.7]. The readers who do not feel comfortable with continuous logic can skip Section 3 with no harm.

Let us note that the converse of Theorem 0.1 does not hold: for example, the theory of a dense circular order is known to be G-compact, but it is not amenable, because it has NIP and $\emptyset$ is not an extension base.

Extreme amenability of automorphism groups of (arbitrary) countable structures M was studied in detail by Kechris, Pestov, and Todorćević. Their paper [18] inspired a whole school, connecting to structural Ramsey combinatorics and dynamics. When $\mathrm{Th}(M)$ is ω-categorical, then extreme amenability of $\mathrm{Aut}(M)$ is a property of this first order theory, so is a model-theoretic notion (in the sense of model theory being the study of first order theories rather than arbitrary structures). Some of this extends to homogeneous models of arbitrary theories and to continuous logic (thanks to Todor Tsankov for a conversation about this with one of the authors).

Let us comment on the relation between extreme amenability of the automorphism group of an ω-categorical, countable structure M as considered in [18] (which we call KPT-extreme amenability) and extreme amenability of $\mathrm{Th}(M)$ in our sense. KPT-extreme amenability concerns *all* flows of the topological group $\mathrm{Aut}(M)$ and says that the universal flow (or rather ambit) has a fixed point. Our first order extreme amenability (of $\mathrm{Th}(M)$) can also be read off from flows of $\mathrm{Aut}(M)$ and says that a *particular flow* $S_{\bar{m}}(M)$ has a fixed point (where $\bar{m}$ is an enumeration of M

and $S_{\bar{m}}(M)$ here denotes the space of complete extensions of $\mathrm{tp}(\bar{m})$ over M). The class of KPT-extremely amenable, ω-categorical theories T is not at present explicitly classified, but appears to be special (perhaps analogous to monadic stability in the stable world). Note that for an (ω-categorical) KPT-extremely amenable theory T, whenever $\mathcal{L}'$ is a countable language extending the language $\mathcal{L}$ of T and T' is a universal $\mathcal{L}'$-theory consistent with T, then the countable model M of T has an expansion to a model of T' in which the new symbols in $\mathcal{L}'$ are interpreted as certain $\emptyset$-definable sets in M. Indeed, such an expansion is just a fixed point of the action of $\mathrm{Aut}(M)$ on the compact (and non-empty) space of the expansions of M to the models of T'. In particular, KPT-extreme amenability of an ω-categorical structure M implies the existence of a $\emptyset$-definable linear ordering on M. By contrast, our first-order extreme amenability is a quite common property: if the Fraïssé limit of a Fraïssé class with free (or, more generally, canonical) amalgamation is ω-categorical, then its theory is extremely amenable (see the discussion after Corollary 2.16); also, every theory T expanded by constants for a model is extremely amenable (here, coheir extensions over this model are the required invariant types); similarly, every stable T expanded by constants for an algebraically closed set in T^{eq} is extremely amenable (non forking extensions witness it by stationary of all types over $\mathrm{acl}^{eq}(\emptyset)$); there are also many NIP or simple theories (e.g. the random graph) which are extremely amenable. Although not explicitly named as extreme amenability, the property of extendibility of types to invariant types has been frequently considered in the literature, and used notably for the elimination of imaginaries; see e.g. [11].

Keisler measures play a big role in this paper (especially in the notion of first order amenability) and we generally assume that the reader is familiar with them. A Keisler measure on a sort (or definable set) X over a model M is simply a finitely additive (probability) measure on the Boolean algebra of definable (over M) subsets of X. As such it is a natural generalization of a complete type over M containing the formula defining X. As pointed out at the beginning of Section 4 of [15], a Keisler measure on X over M is the "same thing" as a regular Borel probability measure on the space $S_X(M)$ of complete types over M containing the formula defining X. Keisler measures are completely natural in model theory, but it took some time for them to be studied systematically. They were introduced in Keisler's seminal paper [19] mainly in a stable environment, and later played an important role in [14] in the solution of some conjectures relating o-minimal groups to compact Lie groups.

This paper contains the material in Section 4 of our preprint "Amenability and definability". Following the advice of editors and referees we have divided that preprint into two papers, the current paper being the second.

1. Preliminaries on G-compactness

We only recall a few basic definitions and facts about Lascar strong types and Galois groups. For more details the reader is referred to [27], [4] or [32].

As usual, by a monster model of a given complete theory we mean a κ-saturated and strongly κ-homogeneous model for a sufficiently large cardinal κ (typically, $\kappa > |T|$ is a strong limit cardinal). Where recall that the (standard) expression "strongly κ-homogeneous" means that any partial elementary map between subsets of the model of cardinality $< \kappa$ extends to an automorphism of the model. A set [tuple] is said to be small [short] if it is of bounded cardinality (i.e. $< \kappa$).

Let $\mathfrak{C}$ be a monster model of a complete theory T.

Definition 1.1.

i) *The group of Lascar strong automorphisms*, which is denoted by $\mathrm{Autf}_L(\mathfrak{C})$, is the subgroup of $\mathrm{Aut}(\mathfrak{C})$ which is generated by all automorphisms fixing a small submodel of $\mathfrak{C}$ pointwise, i.e. $\mathrm{Autf}_L(\mathfrak{C}) = \langle \sigma : \sigma \in \mathrm{Aut}(\mathfrak{C}/M) \text{ for a small } M \prec \mathfrak{C}\rangle$.

ii) *The Lascar Galois group of* T, which is denoted by $\mathrm{Gal}_L(T)$, is the quotient group $\mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_L(\mathfrak{C})$ (which makes sense, as $\mathrm{Autf}_L(\mathfrak{C})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$). It turns out that $\mathrm{Gal}_L(T)$ does not depend on the choice of $\mathfrak{C}$ (e.g. see [27, Fact 4.2]).

The orbit equivalence relation of $\mathrm{Autf}_L(\mathfrak{C})$ acting on any given product S of boundedly (i.e. less than the degree of saturation of $\mathfrak{C}$) many sorts of $\mathfrak{C}$ is usually denoted by E_L. It turns out that this is the finest bounded (i.e. with boundedly many classes), invariant equivalence relation on S (see [20, Proposition 5.4]); and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$ or even to any invariant set. The classes of E_L are called *Lascar strong types*. It turns out that $\mathrm{Autf}_L(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_L-classes on all (possibly infinite) products of sorts. So we see that $\mathrm{Gal}_L(T)$ can be identified with the group of elementary (i.e. induced by $\mathrm{Aut}(\mathfrak{C})$) permutations of all Lascar strong types (as written in the introduction).

For any small $M \prec \mathfrak{C}$ enumerated as $\bar{m}$, we have a natural surjection from $S_{\bar{m}}(M) := \{p \in S(M) : \mathrm{tp}(\bar{m}/\emptyset) \subseteq p\}$ to $\mathrm{Gal}_L(T)$ given by $\mathrm{tp}(\sigma(\bar{m})/M) \mapsto \sigma/\mathrm{Autf}_L(\mathfrak{C})$ for $\sigma \in \mathrm{Aut}(\mathfrak{C})$. We can equip $\mathrm{Gal}_L(T)$ with the quotient topology induced by this surjection, and it is easy to check that this topology does not depend on the choice of M. In this way, $\mathrm{Gal}_L(T)$ becomes a quasi-compact (so not necessarily Hausdorff) topological group (see [32] for a detailed exposition).

Definition 1.2.

i) By $\mathrm{Gal}_0(T)$ we denote the closure of the identity in $\mathrm{Gal}_L(T)$.

ii) *The group of Kim-Pillay strong automorphisms*, which is denoted by $\mathrm{Autf}_{KP}(\mathfrak{C})$, is the preimage of $\mathrm{Gal}_0(T)$ under the quotient homomorphism $\mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_L(T)$.

iii) *The Kim-Pillay Galois group of* T, which is denoted by $\mathrm{Gal}_{KP}(T)$, is the quotient group $\mathrm{Gal}_L(T)/\mathrm{Gal}_0(T) \cong \mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_{KP}(\mathfrak{C})$ equipped with the quotient topology. It is a compact, Hausdorff topological group.

The orbit equivalence relation of $\mathrm{Autf}_{KP}(\mathfrak{C})$ acting on any given product S of (boundedly many) sorts of $\mathfrak{C}$ is usually denoted by E_{KP}. It turns out that this is the finest bounded (i.e. with boundedly many classes), type-definable over $\emptyset$ equivalence relation on S; and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$ (see [27, Lemma 4.18]). The classes of E_{KP} are called *Kim-Pillay strong types*. It turns out that $\mathrm{Autf}_{KP}(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_{KP}-classes on all (possibly infinite) products of sorts. So we see that $\mathrm{Gal}_{KP}(T)$ can be identified with the group of elementary permutations of all Kim-Pillay strong types (as written in the introduction).

The theory T is said to be *G-compact* if the following *equivalent* conditions hold.

(1) $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Autf}_{KP}(\mathfrak{C})$.
(2) $\mathrm{Gal}_L(T)$ is Hausdorff.
(3) Lascar strong types coincide with Kim-Pillay strong types on any (possibly infinite) products of sorts.

Let us briefly explain why the above conditions are equivalent. (1) $\leftrightarrow$ (2) follows from Definitions 1.1 and 1.2. (1) $\rightarrow$ (3) follows from the above definitions of E_L and E_{KP} as the orbit equivalence relations of $\mathrm{Autf}_L(\mathfrak{C})$ and $\mathrm{Autf}_{KP}(\mathfrak{C})$, respectively. Finally, (3) $\rightarrow$ (1) holds, because $\mathrm{Autf}_L(\mathfrak{C})$ and $\mathrm{Autf}_{KP}(\mathfrak{C})$ are the kernels of the actions of $\mathrm{Aut}(\mathfrak{C})$ on the Lascar and Kim-Pillay strong types, respectively.

Lascar's definition of G-compactness from [26] corresponds in our terminology to saying that T remains G-compact after naming any finite set of parameters. Example 2.20 yields a G-compact theory with a non G-compact expansion by a single constant.

By the definition of E_L, we see that $\bar{\alpha}\, E_L\, \bar{\beta}$ if and only if there are $\bar{\alpha}_0 = \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_n = \bar{\beta}$ and models $M_0, \dots, M_{n-1}$ such that

$$\bar{\alpha}_0 \equiv_{M_0} \bar{\alpha}_1 \equiv_{M_1} \dots \bar{\alpha}_{n-1} \equiv_{M_{n-1}} \bar{\alpha}_n.$$

In this paper, by the *Lascar distance* from $\bar{\alpha}$ to $\bar{\beta}$ (denoted by $d_L(\bar{\alpha}, \bar{\beta})$) we mean the smallest natural number n as above. By the *Lascar diameter* of a Lascar strong type $[\bar{\alpha}]_{E_L}$ we mean the supremum of $d_L(\bar{\alpha}, \bar{\beta})$ with $\bar{\beta}$ ranging over $[\bar{\alpha}]_{E_L}$. It is well known (proved in [29]) that $[\bar{\alpha}]_{E_L} = [\bar{\alpha}]_{E_{KP}}$ if and only if the Lascar diameter of $[\bar{\alpha}]_{E_L}$ is finite.

Throughout this paper, tuples of variables are often infinite; in particular, $\varphi(\bar{x})$ means that the free variables of the formula φ are among those listed in $\bar{x}$. It can be convenient to allow $\bar{x}$ to be infinite, though of course φ has only finitely many free variables. By a *finitary* type, we mean a type in finitely many variables. We generally have a fixed underlying complete theory T in the background; a partial type for T can be assumed to include the sentences of T. In any case for partial types $\pi_1(\bar{x})$ and $\pi_2(\bar{x})$, $\pi_1(\bar{x}) \vdash \pi_2(\bar{x})$ means by definition that $T \cup \pi_1(\bar{x})$ logically implies $\pi_2(\bar{x})$.

2. Amenable theories: definitions and basic results

As usual, $\mathfrak{C}$ is a monster model of an arbitrary complete theory T. Let $\bar{c}$ be an enumeration of $\mathfrak{C}$ and let $S_{\bar{c}}(\mathfrak{C}) = \{\mathrm{tp}(\bar{a}/\mathfrak{C}) \in S(\mathfrak{C}) : \bar{a} \equiv \bar{c}\}$. More generally, for a partial type $\pi(\bar{x})$ over $\emptyset$, put $S_\pi(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : \pi \subseteq q\}$. If $p(\bar{x}) \in S(\emptyset)$ and $\bar{\alpha} \models p$, then $S_{\bar{\alpha}}(\mathfrak{C}) := S_p(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : p \subseteq q\}$. (Note that we allow here tuples $\bar{x}$ of unbounded length (i.e. greater than the degree of saturation of $\mathfrak{C}$). Each $S_\pi(\mathfrak{C})$ is naturally an $\mathrm{Aut}(\mathfrak{C})$-flow, i.e. a compact space together with a continuous action of the group $\mathrm{Aut}(\mathfrak{C})$ equipped with the pointwise convergence (equivalently, product) topology.

Let us start from the local version of amenability.

Definition 2.1. A partial type $\pi(\bar{x})$ over $\emptyset$ is *amenable* if there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_\pi(\mathfrak{C})$.

Let μ be a measure as in Definition 2.1. Recall that the restriction of μ to the Baire sets is regular [6, Theorem 7.1.5]. Next, this restriction extends to a unique regular Borel probability measure ν (e.g. see [6, Theorem 7.3.1]). By the construction in the proof of [6, Theorem 7.3.1] and $\mathrm{Aut}(\mathfrak{C})$-invariance of μ, we get that ν is $\mathrm{Aut}(\mathfrak{C})$-invariant. Thus, in Definition 2.1, we can equivalently require a witnessing measure to be *regular* which we usually do.

Remark 2.2. The following conditions are equivalent for a type $\pi(\bar{x})$ over $\emptyset$.

1. $\pi(\bar{x})$ is amenable.
2. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel (regular) probability measure μ on $S_{\bar{x}}(\mathfrak{C})$ concentrated on $S_\pi(\mathfrak{C})$, i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu([\varphi(\bar{x}, \bar{a})]) = 0$ (where $[\varphi(\bar{x}, \bar{a})]$ is the subset of $S_{\bar{x}}(\mathfrak{C})$ consisting of all types containing $\varphi(\bar{x}, \bar{a})$).
3. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on relatively $\mathfrak{C}$-definable subsets of $\pi(\bar{x})$.
4. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on $\mathfrak{C}$-definable sets in variables $\bar{x}$, concentrated on $\pi(\bar{x})$ (i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu(\varphi(\bar{x}, \bar{a})) = 0$).

Proof. Follows easily using the fact (see [8, Proposition 416Q(a)] or [30, Chapter 7.1]) that whenever G acts by homeomorphisms on a compact, Hausdorff, 0-dimensional space X, then each G-invariant, finitely additive probability measure on the Boolean algebra of clopen subsets of X extends to a G-invariant, Borel (regular) probability measure on X. □

Thus, by a *global* $\mathrm{Aut}(\mathfrak{C})$*-invariant Keisler measure extending* $\pi(\bar{x})$ we mean a measure from any of the items of Remark 2.2. And similarly working over any model M in place of $\mathfrak{C}$.

In order to emphasize that a Keisler measure μ is defined on a type space in variables $\bar{x}$, sometimes we will write $\mu_{\bar{x}}$.

Proposition 2.3. *Amenability of a given type $\pi(\bar{x})$ (over $\emptyset$) is absolute in the sense that it does not depend on the choice of the monster model $\mathfrak{C}$. It is also equivalent to the amenability of $\pi(\bar{x})$ computed with respect to an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$.*

Proof. Let M and M' be two $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models. Assume that there is an $\mathrm{Aut}(M)$-invariant, Borel (regular) probability measure μ on $S_\pi(M)$. We want to find such an $\mathrm{Aut}(M')$-invariant measure μ' on $S_\pi(M')$.

Consider any formula $\varphi(\bar{x}, \bar{a}')$ with $\bar{a}' \in M'$. Choose (using the $\aleph_0$-saturation of M) any $\bar{a} \in M$ such that $\bar{a}' \equiv \bar{a}$, and define

$$\mu'([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) := \mu([\varphi(\bar{x}, \bar{a})] \cap S_\pi(M)).$$

By the strong $\aleph_0$-homogeneity of M and $\mathrm{Aut}(M)$-invariance of μ, we see that μ' is well-defined and $\mathrm{Aut}(M')$-invariant. It is also clear that $\mu'(S_\pi(M')) = 1$. It remains to check μ' is finitely additive on clopen subsets (as then μ' extends to the desired Borel measure). Take $\varphi(\bar{x}, \bar{a}')$ and $\psi(\bar{x}, \bar{a}')$ such that $[\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')$ is disjoint from $[\psi(\bar{x}, \bar{a}')] \cap S_\pi(M')$. This just means that $\varphi(\bar{x}, \bar{a}') \wedge \psi(\bar{x}, \bar{a}')$ is inconsistent with $\pi(\bar{x})$. Take $\bar{a} \in M$ such that $\bar{a} \equiv \bar{a}'$. Then $\varphi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{a})$ is still inconsistent with $\pi(\bar{x})$, so the obvious computation using additivity of μ yields: $\mu'(([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) \cup ([\psi(\bar{x}, \bar{a}')] \cap S_\pi(M'))) = \mu'([\varphi(\bar{x}, \bar{a}')] \cap S_\pi(M')) + \mu'([\psi(\bar{x}, \bar{a}')] \cap S_\pi(M'))$. □

Proposition 2.4. *Assume T to be countable, and let $\pi(\bar{x})$ be a partial type. Then $\pi(\bar{x})$ is amenable if and only if for all [sufficiently large] countable, ($\aleph_0$-)homogeneous models M, $\pi(\bar{x})$ has an extension to a Keisler measure $\mu_{\bar{x}}$ over M which is $\mathrm{Aut}(M)$-invariant. If T is uncountable, the same is true but with "countable, $\aleph_0$-homogeneous models" replaced by "strongly $\aleph_0$-homogeneous models of cardinality at most $|T|$".*

Before we prove it, let us explain a few terms from the formulation. By *sufficiently large* models we mean the models from some class $\mathcal{M}$ of models which is closed under isomorphisms and such that for every finitary type $p \in S_n(\emptyset)$ there is a model $M \in \mathcal{M}$ with $p(M) \neq \emptyset$. A model M is said to be *homogeneous* if for every finite tuples $\bar{a} \equiv \bar{b}$ from M and $c \in M$ there is $d \in M$ with $\bar{a}c \equiv \bar{b}d$. If M is countable, this is equivalent to strong $\aleph_0$-homogeneity. For any $A \subset \mathfrak{C}$ of cardinality $\leq |T|$, a standard back-and-forth construction produces a strongly $\aleph_0$-homogeneous model N of cardinality $\leq |T|$ and containing A.

Proof. For each [sufficiently large] countable homogeneous model $M \prec \mathfrak{C}$, let μ_M be an $\mathrm{Aut}(M)$-invariant Keisler measure over M extending $\pi(\bar{x})$, and let $\bar{\mu}_M$ be an arbitrary global Keisler measure extending μ_M. Working in the compact space of global Keisler measures in variables $\bar{x}$, there is a subnet of the net $\{\bar{\mu}_M\}_M$ (with the index set ordered by inclusion), which converges to some $\bar{\mu}$. But then $\bar{\mu}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant: Otherwise, for some formula $\phi(\bar{x},\bar{y})$ and finite tuples $\bar{a},\bar{b}$ in $\mathfrak{C}$ with the same type, we have $\bar{\mu}(\phi(\bar{x},\bar{a}))=r$ and $\bar{\mu}(\phi(\bar{x},\bar{b}))=s$ for some $r<s$. But then we can find some countable homogeneous model M containing $\bar{a},\bar{b}$ and such that $\bar{\mu}_M(\phi(\bar{x},\bar{a}))<\bar{\mu}_M(\phi(\bar{x},\bar{b}))$, contradicting the $\mathrm{Aut}(M)$-invariance of μ_M. □

Lemma 2.5. *A type $\pi(\bar{x})$ (over $\emptyset$) is amenable if and only if each formula $\varphi(\bar{x})$ (without parameters) implied by $\pi(\bar{x})$ is amenable.*

Proof. The implication $(\rightarrow)$ is obvious, as $S_\pi(\mathfrak{C})\subseteq S_\varphi(\mathfrak{C})$, and so for any formula $\psi(\bar{x},\bar{a})$ we can define $\mu'([\psi(\bar{x},\bar{a})]\cap S_\varphi(\mathfrak{C})):=\mu([\psi(\bar{x},\bar{a})]\cap S_\pi(\mathfrak{C}))$, where μ is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_\pi(\mathfrak{C})$.

$(\leftarrow)$. Let Π be the set of formulas in variables $\bar{x}$ which are implied by $\pi(\bar{x})$. By assumption, for every formula $\varphi(\bar{x})\in\Pi$ we can choose an $\mathrm{Aut}(\mathfrak{C})$-invariant Keisler measure μ_φ which is concentrated on $\varphi(\bar{x})$. In the compact space of all Keisler measures in variables $\bar{x}$, there is a subnet of the net $(\mu_\varphi)_{\varphi\in\Pi}$ (with Π ordered by implication of formulas) which converges to some Keisler measure μ. Since all μ_φ are $\mathrm{Aut}(\mathfrak{C})$-invariant, so is μ. In order to see that μ is concentrated on $\pi(\bar{x})$, consider any formula $\psi(\bar{x},\bar{a})$ inconsistent with $\pi(\bar{x})$. Then there is a formula $\varphi(\bar{x})\in\Pi$ inconsistent with $\psi(\bar{x},\bar{a})$. Then every $\theta\in\Pi$ which is implied by $\varphi(\bar{x})$ is also inconsistent with $\psi(\bar{x},\bar{a})$, whence $\mu_\theta(\psi(\bar{x},\bar{a}))=0$. Thus, $\mu(\psi(\bar{x},\bar{a}))=0$. □

Lemma 2.6. *All types in $S(\emptyset)$ (possibly in unboundedly many variables) are amenable if and only if all finitary (i.e. in finitely many variables) types in $S(\emptyset)$ are amenable.*

Proof. The implication $(\rightarrow)$ is trivial. For the other implication, take $p(\bar{x})\in S_{\bar{x}}(\emptyset)$. Consider the compact space $X:=[0,1]^{\{\varphi(\bar{x},\bar{a}):\ \varphi(\bar{x},\bar{y})\text{ a formula, }\bar{a}\in\mathfrak{C}\}}$ with the pointwise convergence topology (where $\bar{x}$ is the fixed tuple of variables). Then the $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on $\mathfrak{C}$-definable sets in variables $\bar{x}$ concentrated on $p(\bar{x})$ form a closed subset $\mathcal{M}$ of X. We can present $\mathcal{M}$ as the intersection of a directed family of closed subsets of X each of which witnessing a finite portion of information of being in $\mathcal{M}$. But each such finite portion of information involves only finitely many variables, so the corresponding closed set is nonempty by the assumption that all finitary types are amenable and Remark 2.2. By the compactness of X, we conclude that $\mathcal{M}$ is nonempty. □

Corollary 2.7. *The following conditions are equivalent.*

1. *All partial types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
2. *All complete types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
3. *All finitary complete types over $\emptyset$ are amenable.*
4. *All consistent formulas (in finitely many variables $\bar{x}$) over $\emptyset$ are amenable.*
5. $\mathrm{tp}(\bar{c}/\emptyset)$ *is amenable (where recall that $\bar{c}$ is an enumeration of $\mathfrak{C}$).*

(6) $\operatorname{tp}(\bar{m}/\emptyset)$ *is amenable for some tuple* $\bar{m}$ *enumerating a model.*

Proof. The equivalence (1) $\leftrightarrow$ (2) is obvious (for (2) $\to$ (1) use the argument as in the proof of ($\to$) in Lemma 2.5). The equivalence (2) $\leftrightarrow$ (3) is Lemma 2.6. The equivalence (3) $\leftrightarrow$ (4) follows from Lemma 2.5. The implications (1) $\to$ (5) $\to$ (6) are trivial. Let us show (6) $\to$ (4). Extend $\bar{m}$ to a tuple $\bar{n}$ also consisting of the elements of M but so that each element of M is repeated infinitely many times. Since the restriction map from $S_{\bar{n}}(\mathfrak{C})$ to $S_{\bar{m}}(\mathfrak{C})$ is an isomorphism of $\operatorname{Aut}(\mathfrak{C})$-flows, by (6), we get that $\operatorname{tp}(\bar{n}/\emptyset)$ is amenable. Hence, $\operatorname{tp}(\bar{n}'/\emptyset)$ is also amenable for every subtuple $\bar{n}'$ of $\bar{n}$. Thus, using Lemma 2.5, we obtain (4), because taking all possible finite subtuples $\bar{n}'$ of $\bar{n}$ and $\varphi(\bar{x}') \in \operatorname{tp}(\bar{n}'/\emptyset)$, we will get all consistent formulas over $\emptyset$ (up to permutations of variables). □

Definition 2.8. The theory T is *amenable* if the equivalent conditions of Corollary 2.7 hold.

By Proposition 2.3, we see that amenability of T is really a property of T, i.e. it does not depend on the choice of $\mathfrak{C}$.

Analogously, one can define the stronger notion of an extremely amenable theory.

Definition 2.9. A type $\pi(\bar{x})$ over $\emptyset$ is *extremely amenable* if there is an $\operatorname{Aut}(\mathfrak{C})$-invariant type in $S_{\pi}(\mathfrak{C})$. The theory T is *extremely amenable* if every type (in any number of variables) in $S(\emptyset)$ is extremely amenable.

As in the case of amenability, compactness arguments easily show that the notions of extremely amenable types and extremely amenable theories are both absolute (i.e. do not depend on the choice of $\mathfrak{C}$), and, in fact, they can be tested on any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model in place of $\mathfrak{C}$; moreover, T is extremely amenable if and only if all finitary types in $S(\emptyset)$ are extremely amenable. Note that Proposition 2.4 specializes to extremely amenable partial types, too. So for countable theories, both amenability and extreme amenability can be seen at the level of countable models. It is also easy to see that in a stable theory, a type in $S(\emptyset)$ is extremely amenable if and only if it is stationary.

Yet another equivalent approach to amenability of T is via $\operatorname{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on the algebra of so-called relatively definable subsets of $\operatorname{Aut}(\mathfrak{C})$. This will be the exact analogue of the definition of definable amenability of definable groups (via the existence of an invariant Keisler measure). We will use this approach in Section 4.

The idea of identifying $\operatorname{Aut}(\mathfrak{C})$ with the subset $\{\sigma(\bar{c}) : \sigma \in \operatorname{Aut}(\mathfrak{C})\}$ of $\mathfrak{C}^{\bar{c}}$ and considering relatively definable subsets of $\operatorname{Aut}(\mathfrak{C})$, i.e. subsets of the form $\{\sigma \in \operatorname{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(\bar{c}), \bar{c})\}$ for a formula $\varphi(\bar{x}, \bar{c})$, already appeared in [25, Appendix A]. Here, we extend this notion of relative definability to the local context and introduce an associated notion of amenability which is easily seen to be equivalent to the amenability of T [or of a certain type in the extended local version].

Let M be any model of T and let $\bar{m}$ be its enumeration.

Definition 2.10. i) By a *relatively definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{m}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, by *relatively $\bar{\alpha}$-definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{\alpha}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.

The above definition differs from the standard terminology in which "A-definable" means "definable over A"; here, "relatively $\bar{\alpha}$-definable" has nothing to do with the parameters over which the set is relatively definable. One should keep this in mind from now on.

For a formula $\varphi(\bar{x}, \bar{y})$ and tuples $\bar{a}$, $\bar{b}$ from M corresponding to $\bar{x}$ and $\bar{y}$, respectively, we will use the following notation

$$A_{\varphi,\bar{a},\bar{b}} = \{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{a}), \bar{b})\}.$$

When $\bar{x}$ and $\bar{y}$ are of the same length (by which we mean that they are also of the same sorts) and $\bar{a} = \bar{b}$, then this set will be denoted by $A_{\varphi,\bar{a}}$.

Note that for any tuple $\bar{\alpha}$ in M, the relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$ form a Boolean $\mathrm{Aut}(M)$-algebra (i.e. a Boolean algebra closed under the action of $\mathrm{Aut}(M)$ by left translations).

Definition 2.11. i) The group $\mathrm{Aut}(M)$ is said to be *relatively amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(M)$.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, the group $\mathrm{Aut}(M)$ is said to be *$\bar{\alpha}$-relatively amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$.

In particular, $\mathrm{Aut}(M)$ being relatively amenable means exactly that it is $\bar{m}$-relatively amenable, where $\bar{m}$ is an enumeration of M.

We will mostly focus on the case when $M = \mathfrak{C}$ is a monster model. But often one can work in the more general context when M is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous, including the case of the unique countable model of an ω-categorical theory.

Proposition 2.12. *Let M be $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous enumerated as $\bar{m}$. Let $\bar{\alpha}$ be a tuple of some elements of M. Then we have:*

1. *The Boolean* $\mathrm{Aut}(M)$*-algebra of clopen subsets of $S_{\bar{\alpha}}(M)$ is isomorphic to the Boolean* $\mathrm{Aut}(M)$*-algebra of relatively $\bar{\alpha}$-definable subsets of* $\mathrm{Aut}(M)$.
2. *The group* $\mathrm{Aut}(M)$ *is $\bar{\alpha}$-relatively amenable if and only if there is an* $\mathrm{Aut}(M)$*-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ (equivalently,* $\mathrm{tp}(\bar{\alpha}/\emptyset)$ *is amenable). In particular,* $\mathrm{Aut}(M)$ *is relatively amenable if and only if there is an* $\mathrm{Aut}(M)$*-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ (equivalently, T is amenable).*

Proof. (1) The assignment $[\varphi(\bar{x},\bar{m})] \mapsto A_{\varphi,\bar{\alpha},\bar{m}}$ clearly defines a homomorphism between the two Boolean $\mathrm{Aut}(M)$-algebras in question (and this does require any assumptions on M). The fact that it is an isomorphism follows easily from $\aleph_0$-saturation and strongly $\aleph_0$-homogeneity of M.

(2) By (1), $\mathrm{Aut}(M)$ is $\bar{\alpha}$-relatively amenable if and only if there is an $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the algebra of clopen subsets of $S_{\bar{\alpha}}(M)$ which in turn is equivalent to the existence of an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$. The fact that the existence of an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ is equivalent to amenability of $\mathrm{tp}(\bar{\alpha}/\emptyset)$ follows from Proposition 2.3. And then, the fact that the existence of an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ is equivalent to amenability of T follows from Corollary 2.7. □

So the terminologies "$\mathrm{Aut}(M)$ is [$\bar{\alpha}$-]relatively amenable" and "T [resp. $\mathrm{tp}(\bar{\alpha}/\emptyset)$] is amenable" will be used interchangeably.

Corollary 2.13. *[For a given tuple $\bar{\alpha}$, $\bar{\alpha}$-]relative amenability of* $\mathrm{Aut}(M)$ *for an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M [containing $\bar{\alpha}$] does not depend on the choice of M.*

The next corollary of Proposition 2.12 will play an essential role Section 4. To state it, we need to extend Definition 2.10 as follows.

Definition 2.14. i) If $\bar{\alpha}$ is a tuple of elements of $\mathfrak{C}$, by a *relatively $\bar{\alpha}$-type-definable* subset of $\mathrm{Aut}(\mathfrak{C})$, we mean a subset of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}),\bar{b}))\}$ for some partial type $\pi(\bar{x},\bar{y})$ (without parameters), where $\bar{x}$ and $\bar{y}$ are short tuples of variables, and $\bar{a}$, $\bar{b}$ are tuples from $\mathfrak{C}$ corresponding to $\bar{x}$ and $\bar{y}$, respectively, such that $\bar{a}$ is a subtuple of $\bar{\alpha}$.
ii) By a *relatively type-definable* subset of $\mathrm{Aut}(\mathfrak{C})$, we mean a relatively $\bar{c}$-type definable subset; equivalently, a subset of $\mathrm{Aut}(\mathfrak{C})$ of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}),\bar{b}))\}$ for some partial type $\pi(\bar{x},\bar{y})$ (without parameters), where $\bar{x}$ and $\bar{y}$ are short tuples of variables, and $\bar{a}$, $\bar{b}$ are corresponding tuples from $\mathfrak{C}$.

We will be using the Boolean algebra generated by all relatively $\bar{\alpha}$-type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$. Observe that this algebra consists of all sets of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/A)) \in \mathcal{P}\}$, where $A \subseteq \mathfrak{C}$ is a (small) set, $\bar{a}$ is a short subtuple of $\bar{\alpha}$, and $\mathcal{P}$ is a finite Boolean combination of closed subsets of $S_{\bar{a}}(A)$.

Corollary 2.15. *Assume $\tilde{\mu}$ is an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(\mathfrak{C})$. For a set $X := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/A)) \in \mathcal{P}\}$ (where $A \subseteq \mathfrak{C}$ is a (small) set, $\bar{a}$ is a short subtuple of $\bar{\alpha}$, and $\mathcal{P}$ is a finite Boolean combination of closed subsets of $S_{\bar{a}}(A)$), put $\mu(X) := \tilde{\mu}(\pi^{-1}[\mathcal{P}])$, where $\pi\colon S_{\bar{\alpha}}(\mathfrak{C}) \to S_{\bar{a}}(A)$ is the restriction map. Then μ is a well-defined,* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively $\bar{\alpha}$-type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$.

In particular, if $\mathrm{Aut}(\mathfrak{C})$ *is relatively amenable, then there exists an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$.

Proof. Easy exercise. □

In the rest of this subsection, we give many examples (or even classes of examples) of amenable and extremely amenable theories.

Recall that a G*-flow* (for a topological group G) is a pair (G, X), where X is a compact, Hausdorff space on which G acts continuously; a G*-ambit* is a G-flow (G, X, x_0) with a distinguished point $x_0 \in X$ with dense G-orbit (e.g. see p. 117-118 of [2] for a discussion on universal ambits). The topological group G is said to be *[extremely] amenable* if each G-flow (equivalently, the universal G-ambit) has an invariant, Borel probability measure [respectively, a fixed point]. (See [10, Chapter III, Theorem 3.1] for several equivalent definitions of amenability for topological groups.)

Corollary 2.16. *Let* M *be* $\aleph_0$*-saturated and strongly* $\aleph_0$*-homogeneous. Then, if* $\mathrm{Aut}(M)$ *is amenable as a topological group (with the pointwise convergence topology), then it is relatively amenable, which in turn implies that it is* $\bar{\alpha}$*-relatively amenable for any tuple* $\bar{\alpha}$ *of elements* M.

Similarly, extreme amenability of $\mathrm{Aut}(M)$ *as a topological group implies extreme amenability of* T.

Proof. Amenability of $\mathrm{Aut}(M)$ implies that there is an $\mathrm{Aut}(M)$-invariant, Borel probability measure on $S_{\bar{m}}(M)$. By Proposition 2.12, this implies relative amenability of $\mathrm{Aut}(M)$. Furthermore, since there is an obvious flow homomorphism from $S_{\bar{m}}(M)$ to $S_{\bar{\alpha}}(M)$, a measure on $S_{\bar{m}}(M)$ induces a measure on $S_{\bar{\alpha}}(M)$, and this is enough by Proposition 2.12. □

As in the introduction, we will call a countable $\aleph_0$-categorical theory KPT-[extremely] amenable if the automorphism group of its unique countable model is [resp. extremely] amenable as a topological group.

So, by Corollary 2.16, KPT-[extreme] amenability of a countable, $\aleph_0$-categorical theory T implies [resp. extreme] amenability of T in the new sense of this paper. In fact most, if not all, of the examples of not only KPT-extremely amenable theories (such as dense linear orderings) but also KPT-amenable theories (such as the random graph, [1, p. 2062]) come from Fraïssé classes with *canonical amalgamation*, hence are extremely amenable in our sense. Only canonical amalgamation over $\emptyset$ is needed here (see Proposition 2.17 below) which says that there is a map $\otimes$ taking pairs of finite structures (A, B) from the Fraïssé class to an amalgam $A \otimes B$ (also in the Fraïssé class) which is compatible with embeddings, i.e. if $f\colon B \to C$ is an embedding of finite structures from the Fraïssé class, then there exists an embedding from $A \otimes B$ to $A \otimes C$ which commutes with f and with the embeddings:

$A \to A \otimes B$, $B \to A \otimes B$, $A \to A \otimes C$, and $C \to A \otimes C$. A typical example is a Fraïssé class with "free amalgamation", namely adding no new relations.

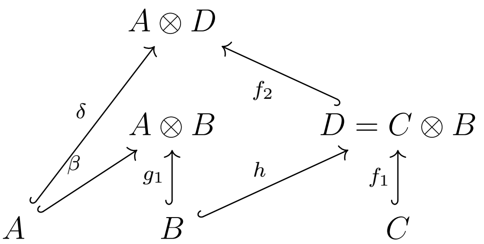
Proposition 2.17. *If M is an ω-categorical structure which is the Fraïssé limit of a Fraïssé class of finite structures in a relational language [or, more generally, finitely generated structures in any language] with canonical amalgamation over $\emptyset$, then $\mathrm{Th}(M)$ is extremely amenable.*

Proof. For simplicity we deal with the case of finite relational structures. All the structures below are from the Fraïssé class in question.

Claim 1: For any finite tuples $\bar{d}, \bar{a}_1, \bar{b}_1, \dots, \bar{a}_n, \bar{b}_n$ from M, if the structures $\bar{a}_i$ and $\bar{b}_i$ are isomorphic (i.e. have the same quantifier-free type), then we can amalgamate structures $\bar{d}$ and $(\bar{a}_i, \bar{b}_i : i \leq n)$ into a structure $\bar{d}', \bar{a}'_1, \bar{b}'_1, \dots, \bar{a}'_n, \bar{b}'_n$ in such a way that $\bar{a}'_i$ is isomorphic with $\bar{b}'_i$ over $\bar{d}'$ for all $i \leq n$.

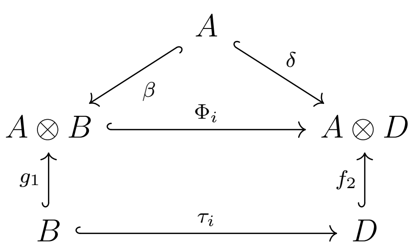
Proof. Let A be the substructure of M consisting of the coordinates of the tuple $\bar{d}$, B the substructure of M consisting of the coordinates of the tuples $\bar{a}_1, \dots, \bar{a}_n$, and C a substructure of M containing the coordinates of the tuples $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n$ and such that for every i the isomorphism $\bar{a}_i \mapsto \bar{b}_i$ extends to an embedding $\sigma_i \colon B \to C$. Let $D := C \otimes B$.

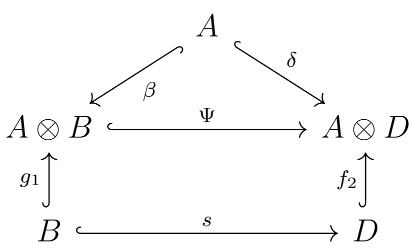
We have the following collection of canonical embeddings:



Let $f := f_2 \circ f_1 \colon C \to A \otimes D$ and $\tau_i := f_1 \circ \sigma_i \colon B \to D$. Let $s \colon B \to D$ be the embedding given by $s(\bar{a}_i) := f_1(\bar{a}_i)$ for all i.

By canonical amalgamation, we have the following commutative diagrams of embeddings:





We claim that $\bar{a}_i' := f(\bar{a}_i)$, $\bar{b}_i' := f(\bar{b}_i)$, and $\bar{d}' := \delta(\bar{d})$ are as required. It is clear that $(\bar{a}_i, \bar{b}_i : i \leq n)$ is isomorphic to $(\bar{a}_i, \bar{b}_i : i \leq n)$, and $\bar{d}$ to $\bar{d}'$. The fact that $\bar{a}_i' \equiv^{qf}_{\bar{d}'} \bar{b}_i'$ can be seen via the following computation based on the above diagrams: $\Phi_i(\Psi^{-1}(\bar{a}_i')) = \Phi_i(\Psi^{-1}(f_2(f_1(\bar{a}_i)))) = \Phi_i(\Psi^{-1}(f_2(s(\bar{a}_i)))) = \Phi_i(g_1(\bar{a}_i)) = f_2(\tau_i(\bar{a}_i)) = f_2(f_1(\sigma_i(\bar{a}_i))) = f(\bar{b}_i) = \bar{b}_i'$, and $\Phi_i(\Psi^{-1}(\bar{d}')) = \Phi_i(\Psi^{-1}(\delta(\bar{d}))) = \Phi_i(\beta(\bar{d})) = \delta(\bar{d}) = \bar{d}'$. □(claim)

By the claim, using ω-categoricity and quantifier elimination, one concludes by compactness that any finitary type in $S(\emptyset)$ extends to an $\mathrm{Aut}(M)$-invariant type in $S(M)$, so T is extremely amenable (since M is ω-categorical). □

In [24], we proved that both KPT-amenability and KPT-extreme amenability are preserved by adding finitely many parameters. This is not the case for our notion of first order [extreme] amenability as shown by the following two examples. Before that observe that if T is [extreme] amenability, then so is T^{eq}.

Example 2.18. Let T be the theory of two equivalence relations E_1, E_2, where E_1 has infinitely many classes, all infinite, and each E_1-class is divided into two E_2-classes, both infinite. Then T is extremely amenable, but adding an (imaginary) parameter for an E_1-class destroys extreme amenability.

Proof. Using a standard back-and-forth argument, one checks that T has quantifier elimination. To show extreme amenability of T, consider any type $\pi(\bar{x})$ without parameters. It is clear (using quantifier elimination) that $\pi(\bar{x})$ extends to $p \in S_{\bar{x}}(\mathfrak{C})$ so that $\neg E_1(x_i, c) \in p$ for all coordinates x_i of $\bar{x}$ and all $c \in \mathfrak{C}$. Then also $\neg E_2(x_i, c) \in p$ for all x_i and c as before. By quantifier elimination, p is clearly $\mathrm{Aut}(\mathfrak{C})$-invariant.

Now, add a constant for an E_1-class C. Consider the formula (over $\emptyset$) $\varphi(x) := (x \in C)$. Let $p \in S_x(\mathfrak{C})$ be any global type containing $\varphi(x)$. Then p determines one of the two E_2-classes into which C is divided by E_2. But, by quantifier elimination, there is an automorphism of $\mathfrak{C}$ which swaps these two classes, so moves p, and hence p is not $\mathrm{Aut}(\mathfrak{C})$-invariant. □

Example 2.19. Let T be the theory of an equivalence relation E with infinitely many infinite classes and a ternary relation $S(x, y, z)$ which is exactly the disjoint union of dense circular orders on all the E-classes. Then T is extremely amenable, but adding an (imaginary) parameter for an E-class destroys even amenability.

Proof. Again we have quantifier elimination, and extremely amenability can be seen as in the last example. Now, add a constant for an E-class C. Consider the formula (over $\emptyset$) $\varphi(x) := (x \in C)$. By quantifier elimination, the structure induced on C is interdefinable over $\emptyset$ with a monster model $\mathfrak{C}'$ of the theory of a dense circular order, and after dividing by the kernels K and K' of the relevant actions, the $\mathrm{Aut}(\mathfrak{C})/K$-flow $S_\varphi(\mathfrak{C})$ is isomorphic to the $\mathrm{Aut}(\mathfrak{C}')/K'$-flow $S_1(\mathfrak{C}')$. Since the latter flow does not carry an invariant, Borel probability measure (by

Remark 2.25 below, because the formula $x = x$ forks over $\emptyset$ in dense circular orders), neither does the former one. □

We take the opportunity and modify the above example to get a G-compact theory whose expansion by an imaginary parameter is not G-compact.

Example 2.20. Let T_0 be any single-sorted non G-compact theory (e.g. from [4, Proposition 4.5]) in a language L_0 without constants. Let T be the theory of an equivalence relation E with infinitely many classes, and each class has the L_0-structure of a model of T_0 (and for any n-ary function symbol f and a tuple $\bar{a} = (a_0, \dots, a_{n-1})$ containing elements from at least 2 different E-classes, we have $f(\bar{a}) = a_0$). Then T is G-compact (even G-trivial), but G-compactness is destroyed by adding an (imaginary) parameter for an E-class.

Proof. By a back-and-forth argument, T has quantifier elimination relative to the Morleyization of T_0. Recall that $\mathfrak{C} \models T$ is a monster model. It is a disjoint union of monster models of T_0 on all E-classes. Note that any union of infinitely many E-classes is a model of T, in fact, an elementary substructure of $\mathfrak{C}$. Using this observation, one easily gets that any two tuples of bounded length with the same type over $\emptyset$ have the same type over a model, i.e. they are at Lascar distance at most 1. Thus, T is G-trivial. On the other hand, take any tuple $\bar{a}$ contained in a single E-class C and such that $[\bar{a}]_{E_L} \neq [\bar{a}]_{E_{KP}}$ in the sense of T_0 (working in $C \models T_0$). Then $[\bar{a}]_{E_L} \neq [\bar{a}]_{E_{KP}}$ in the sense of T expanded by a constant for the class C, which follows from the obvious observation that $\mathrm{Aut}^{T_0}(C)$ coincides with $\mathrm{Aut}(\mathfrak{C}/(\mathfrak{C} \setminus C))|_C$ and the result from [29] saying that $[\bar{a}]_{E_L} \neq [\bar{a}]_{E_{KP}}$ if and only if $[\bar{a}]_{E_L}$ has infinite Lascar diameter. □

Let us look at Example 2.20 from the perspective of the main result of this paper. By an argument as in Example 2.18, the theory T from Example 2.20 is extremely amenable. Thus, G-triviality alternatively follows from Proposition 4.2. On the other hand, since after adding a constant the theory is not G-compact, it is also nonamenable by Theorem 0.1.

Example 2.21. Let M_n be the unit circle equipped with the ternary relation S_n of circular order and with the clockwise rotation g_n of order n, and let $T_n := \mathrm{Th}(M_n)$ (see [4, Section 4]); the language of T_n, consisting of S_n and g_n, will be denoted by L_n. Let T^- be the theory of an equivalence relation E and pairwise disjoint unary predicates P_n, $n \in \omega$, where each P_n is a union of infinitely many E-classes, and each E-class on P_n has the L_n-structure of a model of T_n (and $g_n(a) = a$ for $a \notin P_n$). Finally, let T be the theory T^- canonically expanded by the additional sort of all E-classes (but we do not add anything else from T^{eq}). Then T is G-compact after naming any finitely many parameters, but is not G-compact after naming countably many elements a_n/E, $n < \omega$ (where $a_n \in P_n$ are chosen arbitrarily).

Proof. By a back-and-forth argument, T has quantifier elimination. For the monster model $\mathfrak{C}$ of T we have that each E-class on P_n is a monster model of T_n.

Again, any union of E-classes which contains infinitely many E-classes on each P_n is an elementary substructure of $\mathfrak{C}$. Using this observation, the fact that T is G-compact after naming any finitely many parameters follows from the observation that each theory T_n has this property (which we leave as an exercise).

Now, consider any tuple $\bar{a} = (a_n)_{n\in\omega}$ with $a_n \in P_n$. Add constants for all the classes a_n/E, $n \in \omega$, and denote the resulting expansion of T by T^c. Then $[a_n]_{E_L}$ in the sense of T_n (working in $[a_n]_E \models T_n$) coincides with $[a_n]_{E_L}$ in the sense of T^c, and the Lascar diameters of $[a_n]_{E_L}$ in the sense of both theories also agree. By [4, Corollary 4.4], this diameter is greater than $n/2$. Thus, the diameter of $[\bar{a}]_{E_L}$ (in T^c) is infinite, so T^c is not G-compact. □

The next fact follows easily from [15, Proposition 2.11].

Fact 2.22. *In an NIP theory, for any global type p the following conditions are equivalent:*

(1) p does not fork over $\emptyset$.
(2) The $\mathrm{Aut}(\mathfrak{C})$-orbit of p is bounded.
(3) p is Kim-Pillay invariant (i.e. invariant under $\mathrm{Autf}_{KP}(\mathfrak{C})$).
(4) p is Lascar invariant.

More importantly, Proposition 4.7 of [15] can be stated as:

Fact 2.23. *In an NIP theory, a type $p \in S(\emptyset)$ is amenable if and only if it does not fork over $\emptyset$ (equivalently, it has a global non-forking extension).*

Although Fact 2.23 is proved in [15], let us give a sketch of the proof of ($\leftarrow$) in order to see how the desired measure is obtained. So assume that $p(\bar{x})$ does not fork over $\emptyset$, and take its global non-forking extension $q(\bar{x}) \in S_{\bar{x}}(\mathfrak{C})$. Take $\bar{\alpha} \models p$. Consider any formula $\varphi(\bar{x}, \bar{b})$. Recall that

$$A_{\varphi,\bar{\alpha},\bar{b}} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(\bar{\alpha}), \bar{b})\}.$$

Let $S_\varphi := \{\bar{b}' : \varphi(\bar{x}, \bar{b}') \in q\}$. By Fact 2.22, q is $\mathrm{Autf}_{KP}(\mathfrak{C})$-invariant. So, by the argument in Proposition 2.6 of [15] and NIP, there is $N < \omega$ such that

$$S_\varphi = \bigcup_{n<N} A_n \cap B_n^c,$$

where each A_n and B_n is type-definable and invariant under $\mathrm{Autf}_{KP}(\mathfrak{C})$. Let

$$\tilde{S}_{\varphi(\bar{x},\bar{b})} := \{\sigma/\mathrm{Autf}_{KP}(\mathfrak{C}) : \varphi(\bar{x}, \sigma^{-1}(\bar{b})) \in q\} = \{\sigma/\mathrm{Autf}_{KP}(\mathfrak{C}) : \varphi(\bar{x}, \bar{b}) \in \sigma(q)\}.$$

Using the above formula for S_φ, one shows that $\tilde{S}_{\varphi(\bar{x},\bar{b})}$ is a Borel (even constructible) subset of $\mathrm{Gal}_{KP}(T)$.

Let $\mathfrak{h}$ be the unique (left invariant) normalized Haar measure on the compact group $\mathrm{Gal}_{KP}(T)$. By the last paragraph, $\tilde{S}_{\varphi(\bar{x},\bar{b})}$ is Borel, hence $\mathfrak{h}(\tilde{S}_{\varphi(\bar{x},\bar{b})})$ is defined, and so we can put

$$\mu(A_{\varphi,\bar{\alpha},\bar{b}}) := \mathfrak{h}(\tilde{S}_{\varphi(\bar{x},\bar{b})}).$$

It is easy to check that μ is a well-defined (i.e. does not depend on the choice of φ yielding the fixed set $A = A_{\varphi,\bar{\alpha},\bar{b}}$), $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(\mathfrak{C})$. Thus, $\mathrm{Aut}(\mathfrak{C})$ is $\bar{\alpha}$-relatively amenable; equivalently, p is amenable.

By Fact 2.23 and [15, Corollary 2.14], we have

Corollary 2.24. *Assume T has NIP. Then, T is amenable if and only if $\emptyset$ is an extension base (i.e. any type over $\emptyset$ does not fork over $\emptyset$). In particular, stable, o-minimal, and c-minimal theories are all amenable (even after adding constants).*

In the above proof of the implication ($\leftarrow$) in Fact 2.23, NIP plays an essential role to get that q is $\mathrm{Autf}_{KP}(\mathfrak{C})$-invariant and that $\tilde{S}_\varphi$ is Borel. On the other hand, the implication ($\rightarrow$) in Fact 2.23 is completely general. Namely, we have

Remark 2.25. In an arbitrary theory, if a partial type $\pi(\bar{x})$ over $\emptyset$ is amenable, then it does not fork over $\emptyset$. In particular, in an arbitrary amenable theory, $\emptyset$ is an extension base.

Proof. Let μ be a global, invariant Keisler measure extending $\pi(\bar{x})$. Choose a μ-wide type $q(\bar{x}) \in S_{\bar{x}}(\mathfrak{C})$ (i.e. any formula in $q(\bar{x})$ is of positive measure). Then, one easily checks that $q(\bar{x})$ does not fork over $\emptyset$, so we are done. □

Thus, amenability of T is a strong form of saying that $\emptyset$ is an extension base; and amenability after adding any constants is a strong form of saying that every set is an extension base.

By [15, Corollary 2.10], the characterization from Corollary 2.24 gives us

Corollary 2.26. *Assume T has NIP. Then amenability of T implies G-compactness.*

Theorem 0.1 is a generalization of the last corollary to arbitrary amenable theories, but it requires completely different methods compared with the NIP case.

It is worth mentioning that Theorem 7.7 of [23] yields several other conditions equivalent (under NIP) to the existence of $p \in S_{\bar{c}}(\mathfrak{C})$ with bounded $\mathrm{Aut}(\mathfrak{C})$-orbit (and so to amenability of T), for example: some (equivalently, every) minimal left ideal of the Ellis semigroup of the $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c}}(\mathfrak{C})$ is of bounded size. In particular, a variant of Newelski's conjecture proved in [23, Theorem 0.7] can be stated as follows: if T is an amenable theory with NIP, then a certain natural epimorphism from the Ellis group of T to $\mathrm{Gal}_{KP}(T)$ is an isomorphism. This also implies G-compactness of amenable, NIP theories.

Let us mention in this section some relations between our notions of amenability and extreme amenability of a theory T and the notion of a strongly determined over $\emptyset$ theory from [16] (originating in work of Ivanov and Macpherson [17]). Decoding the definition in [16], T is strongly determined over $\emptyset$ if any complete type $p(\bar{x})$ over $\emptyset$ has an extension to a complete type $p'(\bar{x})$ over $\mathfrak{C}$ which is $\mathrm{acl}^{eq}(\emptyset)$-invariant. So clearly T extremely amenable implies T is strongly determined over

$\emptyset$. Moreover, by Corollary 2.24, assuming NIP, T strongly determined over $\emptyset$ implies amenability of T. In fact, if T is NIP and KP-strong types agree with usual strong types (over $\emptyset$), then T is strongly determined over $\emptyset$ iff T is amenable.

We finish this section with a list of some examples of classes of [extremely] amenable theories.

- By Corollary 2.16, every countable, ω-categorical structure with [extremely] amenable group of automorphisms has [extremely] amenable theory. Many concrete examples of such structures were found in [18] and in later papers which further studied "KPT theory". For example, the theory of any ordered random hypergraph is extremely amenable.
- By Proposition 2.17, whenever a Fraïssé class with canonical amalgamation over $\emptyset$ has ω-categorical Fraïssé limit, then the theory of this limit is extremely amenable. For example, the theory of any random hypergraph is extremely amenable.
- By Corollary 2.24, all NIP theories for which $\emptyset$ is an extension base are amenable; in particular, all stable, o-minimal, and C-minimal theories are amenable. But o-minimal theories are even extremely amenable, because any global non-forking extension of a given type over $\emptyset$ is $\mathrm{Autf}_L(\mathfrak{C})$-invariant by Fact 2.22, and, on the other hand, in o-minimal theories, $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Aut}(\mathfrak{C})$ by [32, Lemma 24]. A stable theory is extremely amenable if and only of all [finitary] complete types over $\emptyset$ are stationary.
- The theories of all measurable structures in the sense of Elwes and Macpherson (e.g. pseudo-finite fields, smoothly approximable structures) are amenable by [7, Remark 3.8(5)] and Corollary 2.7(4). Those theories are supersimple of finite D-rank by Corollaries 3.6 and 3.7 of [7].
- Let $\mathcal{C}$ be a class of finite structures of size at least 2 in a language L containing constants c_1, c_2 interpreted as distinct elements in all structures in $\mathcal{C}$. Then the structures $A \in \mathcal{C}$ can be canonically expanded to L'-structures A' with $\mathrm{Aut}(A') = \mathrm{Aut}(A)$, such that letting $\mathcal{C}' := \{A' : A \in \mathcal{C}\}$, $T' := \mathrm{Th}(\mathcal{C}')$ is amenable in the sense that all its completions are amenable; and remains so over any finite set. (It suffices to close the language of $\mathcal{C}$ under cardinality comparison quantifiers $Q\bar{x}\bar{y}(\phi(\bar{x},\bar{u}), \psi(\bar{y},\bar{v}))$, asserting in finite models that there are at least as many tuples $\bar{x}$ with $\phi(\bar{x},\bar{u})$ as tuples $\bar{y}$ with $\psi(\bar{y},\bar{v})$, where $\bar{x},\bar{u},\bar{y},\bar{v}$ are finite tuples of variables; see the first page of [5, Section 8.3]. (Though the general quantifiers referred to here are actually only briefly mentioned before passing to a more specialized and more effective version appropriate there.) It is easy to see (using constants c_1, c_2) that using these quantifiers, one can also express the relation $p|D(\bar{a})| \geq q|D'(\bar{b})|$ as $R_{p,q,D,D'}(\bar{a},\bar{b})$ for some L'-formula $R_{p,q,D,D'}(\bar{x},\bar{y})$ (without extra parameters), where p, q are positive integers and D, D' are definable families of definable sets. Hence, for any $\bar{a}$-definable set $D = D(\bar{a})$ in a monster model

$\mathfrak{C}$ of any completion of T', we can define an $\mathrm{Aut}(\mathfrak{C}/\bar{a})$-invariant Keisler measure on definable subsets of D by: $\mu(E(\bar{b})) := \inf\{p/q : \mathfrak{C} \models R_{p,q,D,E}(\bar{a}, \bar{b})\}$. Thus, the theory T' is amenable. In fact, it is even definably amenable in the sense of Definition 3.1 below.)

- In [21] (which was written about a year after the original preprint containing the material from this paper), Ramsey-theoretic characterizations of [extreme] amenability as well as various other dynamical properties of first order theories are established. Also, examples of amenable theories illustrating some other important phenomena (which we will not mention here) are given there. Let us only say that e.g. Examples 5.10 and 5.11 from that paper yield some amenable theories which are not extremely amenable, not NIP, and supersimple of SU-rank 1. Expanding them by an "independent" dense linear order, we get examples of amenable theories which are not extremely amenable, not NIP, and not simple. Namely, one can easily show the following.

  Let T be the theory in a relational language $\{R, E_n, \leq\}_{n<\omega}$ (where all symbols are binary) saying that R is irreflexive and symmetric, each E_n is an equivalence relation with at least two classes and E_0 has finitely many classes, $\leq$ is a dense linear order, and the relations R, $\{E_n\}_{n<\omega}$, $\leq$ are "independent" in the sense that the intersection of
  (1) any finite collection of formulas of the form $\pm R(x, a)$ (with pairwise distinct a's) with
  (2) any finite collection $C_0, \ldots, C_{n-1}$ of classes of the relations $E_0, \ldots, E_{n-1}$, respectively, with
  (3) any open $\leq$-interval

  is non-empty. Then T is complete with quantifier elimination, amenable (as in the proof of amenability in [21, Example 5.10], and see also the next bullet), but not extremely amenable (because each global 1-type determines an E_0-class which can be moved to another E_0-class by some automorphism), not NIP (because the reduct to R is the random graph), and not simple (because of the dense linear order $\leq$).
- One can extend the last example as follows. Let us start from any amenable theory T_0 for which $\mathrm{acl}(A) = A$ for every A (in place of the theory of independent equivalence relations $\{E_n\}_n$). Then add independently a random graph and a dense linear order to obtain a new theory T (by "independently" we mean that the intersection of any infinite definable set with any collection of finitely many formulas of the form $\pm R(x, a)$ (with pairwise distinct a's) and with any open interval is non-empty). Then T is amenable, non-NIP, and non-simple.

  The idea of the proof is as follows. We may assume that T_0 has q.e (by taking Morleyzation), and then so does T (by a back-and-forth argument, using randomness and the assumption that $\mathrm{acl}(A) = A$). Take any finitary type $p(\bar{x}) \in S^T_{\bar{x}}(\emptyset)$ (in the theory T), where $\bar{x} = (x_i)_{i<m}$; let $p_0 \in S^{T_0}_{\bar{x}}(\emptyset)$ be its restriction in the theory T_0. Let $\mathfrak{C}$ be a monster model of both T and T_0. Consider the partial type $\pi(\bar{x}) := \{x_i \geq c : i < m, c \in \mathfrak{C}\} \cup \{R(x_i, a) : i < m, a \in \mathfrak{C}\}$ in T, and let $X := [\pi(\bar{x})]$ be the induced closed subset of $S^T_p(\mathfrak{C})$. By randomness and the extra assumption on acl, X is non-empty, so it is an $\mathrm{Aut}^T(\mathfrak{C})$-flow. Let $\Phi \colon S^T_p(\mathfrak{C}) \to S^{T_0}_{p_0}(\mathfrak{C})$ be the restriction map. By q.e., randomness, and the assumption on acl, one easily checks that $\Phi|_X \colon X \to S^{T_0}_{p_0}(\mathfrak{C})$ is a monomorphism of $\mathrm{Aut}^T(\mathfrak{C})$-flows whose image is the $\mathrm{Aut}^{T_0}(\mathfrak{C})$-subflow Y of $S^{T_0}_{p_0}(\mathfrak{C})$ given by $\{x_i \neq c : i < m, c \in \mathfrak{C}\}$. Since T_0 is amenable, Y carries an $\mathrm{Aut}^{T_0}(\mathfrak{C})$-invariant, Borel probability measure. The pullback of this measure under $\Phi|_X$ is an $\mathrm{Aut}^T(\mathfrak{C})$-invariant, Borel probability measure on X, which witnesses that $p(\bar{x})$ is amenable. Thus, T is amenable.

3. Amenability implies G-compactness: the case of definable measures

Theorem 0.1 will be proved in full generality in Section 4. However, some special cases have a relatively easy proof. One such is the NIP case above. Another case is when T is extremely amenable, where the proof of Remark 4.21 of [24] shows that in fact T is G-trivial (the Lascar group is trivial). This is made explicit in Proposition 4.2 below. Ivanov's observation in [16] that if T is strongly determined over $\emptyset$, then Lascar strong types coincide with (Shelah) strong types follows from Proposition 4.2 by working over $\mathrm{acl}^{eq}(\emptyset)$. However, deducing G-compactness of T from amenability of T in general is more complicated, and the proof in Section 4 uses a version of the stabilizer theorem (i.e. Corollary 2.12 of [13]) and requires adaptations of some ideas from Section 2 of [13] involving various computations concerning relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$. This section is devoted to a proof of the main result in the special case when amenability of T is witnessed by $\emptyset$-definable, global Keisler measures, rather than just $\emptyset$-invariant Keisler measures (see Definition 3.1 below). We will make use of continuous logic stability as in Section 3 of [13]. We want to clarify that we are not trying to give here an introduction to continuous logic in the sense of the precise formalism of [3]. In [13], we gave a self contained account of a certain approach to continuous logic in the context of classical first order theories. Here, we discuss, among other things, compatibilities of our approach in [13] with the specific formalism of [3], as we want to make explicit use of results from [3].

Recall the standard notion of a definable function from a model to a compact, Hausdorff space (the equivalent statements given below follow from [9, Lemma 3.2]). A function $f \colon M^n \to C$ (where M is a model and C is a compact, Hausdorff space) is called *definable* if the preimages under f of any two disjoint closed subsets of C can be separated by a definable subset of M^n; equivalently, f is induced by a (unique) continuous map from $S_n(M)$ to C. This is equivalent to the condition that f has a (unique) extension to an M-*definable* function $\hat{f}\colon \mathfrak{C}^n \to C$ (where $\mathfrak{C}$ is a monster model), meaning that the preimages under $\hat{f}$ of all closed subsets of C are type-definable over M. A function from $\mathfrak{C}^n$ to C is said to be A-definable, if the preimages of all closed subsets are type-definable over A. In particular, a Keisler measure $\mu(\bar{x})$ is said to be $\emptyset$-*definable* if for every formula $\varphi(\bar{x}, \bar{y})$, the function $\mu(\varphi(\bar{x}, \bar{y})) : \mathfrak{C}^{\bar{y}} \to [0, 1]$ is $\emptyset$-definable.

Definition 3.1. A theory T is *definably amenable* if every formula $\varphi(\bar{x})$ extends to a $\emptyset$-definable, global Keisler measure.

We are aware that, while there is no formal clash, this is a different use of the adverb than in the case of definable groups; "definably" there refers to the measure algebra, not to the measure . Thus "definably definably amenable" would express, in addition, that the measure μ on $\mathrm{Def}(G)$ is itself definable. In the case of theories, the inner qualifier is redundant.

We first discuss the relationship between our formalism from Section 3 of [13] and that of [3]. Start with our (classical) complete first order theory T, which we assume for convenience to be 1-sorted. This is a theory in continuous logic in the sense of [3], but where the metric is discrete and all relation symbols are $\{0, 1\}$ valued, where 0 is treated as "true" and 1 as "false". The type spaces $S_n(T)$ are of course Stone spaces. Recall from Definition 3.4 of [13] that by a *continuous logic (CL) formula over* A we mean a continuous function $\phi\colon S_n(A) \to \mathbb{R}$. If ϕ is such a CL-formula, then for any $\bar{b} \in M^n$ (where $M \models T$) by $\phi(\bar{b})$ we mean $\phi(\mathrm{tp}(\bar{b}/A))$. So CL-formulas over A can be thought of as A-definable maps from $\mathfrak{C}^n$ to compact subsets of $\mathbb{R}$ (note that the range of every CL-formula is compact). What are called *definable predicates*, in finitely many variables and without parameters, in [3] are precisely CL-formulas over $\emptyset$ in our sense, but where the range is contained in $[0, 1]$. Namely, a definable predicate in n variables is given by a continuous function from $S_n(T)$ to $[0, 1]$. The CL-generalization of Morleyizing T consists of adding all such definable predicates as new predicate symbols in the sense of continuous logic. So if M is a model of T and $\phi(\bar{x})$ is such a new predicate symbol, then the interpretation $\phi(M)$ of ϕ in M is the function taking an n-tuple $\bar{a}$ from M to $\phi(\mathrm{tp}(\bar{a}))$. Let us call this new theory T_{CL} (a theory of continuous logic), to which we can apply the results of [3]. By the discussion after Proposition 3.10 in [3], one sees that T_{CL} has quantifier elimination [3, Definition 4.14]. As just remarked, any model M of T expands uniquely to a model of T_{CL}, but we will still call it M. Note also that any saturation or homogeneity property of M is preserved under passing to T_{CL} (which follows from the observations that the group of automorphisms of M is preserved and types in $S^T(A)$ determine types in $S^{T_{CL}}(A)$ for any A).

To understand *imaginaries* as in Section 5 of [3], we have to also consider definable predicates, without parameters, but in possibly infinitely (yet countably) many variables. As in Proposition 3.10 of [3], such a definable predicate in infinitely many variables can be identified with a continuous function from $S_\omega(T)$ to $[0,1]$, where $S_\omega(T)$ is the space of complete types of T in a fixed countable sequence of variables. We feel free to call such a function (and the corresponding function on ω-tuples in models of T to $[0,1]$) a CL-formula in infinitely many variables. Let us now fix a definable predicate (so CL-formula) $\phi(\bar{x},\bar{y})$, where $\bar{x}$ is a finite tuple of variables, and $\bar{y}$ is a possibly infinite (but countable) sequence of variables. A "code" for the CL-formula (with parameters $\bar{a}$ and finite tuple $\bar{x}$ of free variables) $\phi(\bar{x},\bar{a})$ will then be a CL-imaginary in the sense of [3], and all CL-imaginaries will arise in this way. The precise formalism (involving new sorts with their own distance relation) is not so important, but the point is that the code will be something fixed by precisely those automorphisms (of a saturated model) which fix the formula $\phi(\bar{x},\bar{a})$. More precisely, the code will be the equivalence class of $\bar{a}$ with respect to the obvious equivalence relation $E_\phi(\bar{y},\bar{z})$, on tuples of the appropriate length. If $\bar{y}$ is a finite tuple of variables, then we will call a corresponding imaginary (i.e. code for $\phi(\bar{x},\bar{a})$) a *finitary CL-imaginary.* We will work in the saturated model $\bar{M}=\mathfrak{C}$ of T which will also be a saturated model of T_{CL}. When we speak about interdefinability of various objects, we mean a priori in the sense of automorphisms of $\bar{M}$, i.e. two objects are *interdefinable* if they are preserved by exactly the same automorphisms of $\bar{M}$.

The notion of hyperimaginaries is well-established in (usual, classical) model theory [27]. A *hyperimaginary* is by definition $\bar{a}/E$, where $\bar{a}$ is a possibly infinite (but small compared with the saturation) tuple and E a type-definable over $\emptyset$ equivalence relation on tuples of the relevant size. Up to interdefinability we may restrict to tuples of length at most ω (see [31, Remark 3.1.8]). When the length of $\bar{a}$ is finite, we call $\bar{a}/E$ a *finitary hyperimaginary.*

Remark 3.2. If E is a type-definable over $\emptyset$ equivalence relation and the $\mathrm{Aut}(\bar{M})$-orbit of $\bar{a}/E$ is bounded, then there is a *bounded* type-definable over $\emptyset$ equivalence relation F refining $\equiv$ which agrees with E on $[\bar{a}]_\equiv$.

Proof. By [27, Lemma 4.18], $F:=E_{KP}\cup(E\cap([\bar{a}]_\equiv\times[\bar{a}]_\equiv))$ works, because $E\cap([\bar{a}]_\equiv\times[\bar{a}]_\equiv)$ is $\emptyset$-type-definable and bounded by assumption. (More precisely, [27, Lemma 4.18] is stated for finite tuples but works the same for infinite tuples, too). □

The following is routine, but we sketch the proof.

Lemma 3.3. *(i) Any [finitary] CL-imaginary is interdefinable with a [finitary] hyperimaginary.*
(ii) If E is a bounded, type-definable over $\emptyset$ equivalence relation, then each class of E is interdefinable with a sequence of finitary CL-imaginaries.

Proof. (i) If $\phi(\bar{x},\bar{y})$ is a CL-formula where $\bar{y}$ is a possibly countably infinite tuple, then the equivalence relation $E(\bar{y},\bar{z})$ which says of $(\bar{b},\bar{c})$ that the functions

$\phi(\bar{x}, \bar{b})$ and $\phi(\bar{x}, \bar{c})$ are the same is a type-definable over $\emptyset$ equivalence relation in T. (Indeed, consider any $\bar{b}, \bar{c}$ such that $E(\bar{b}, \bar{c})$ does not hold, i.e. for some $\bar{a}$, $\phi(\bar{a}, \bar{b}) \neq \phi(\bar{a}, \bar{c})$. Then there are formulas $\varphi(\bar{x}, \bar{y}) \in \mathrm{tp}(\bar{a}\bar{b}/\emptyset)$ and $\psi(\bar{x}, \bar{y}) \in \mathrm{tp}(\bar{a}\bar{c}/\emptyset)$ such that whenever $\models \varphi(\bar{a}', \bar{b}') \wedge \psi(\bar{a}', \bar{c}')$, then $\phi(\bar{a}', \bar{b}') \neq \phi(\bar{a}', \bar{c}')$. Put $\theta(\bar{y}, \bar{z}) := \exists \bar{x} \varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{z})$. Then $\models \theta(\bar{b}, \bar{c})$ and whenever $\models \theta(\bar{b}', \bar{c}')$, then $\phi(\bar{a}', \bar{b}') \neq \phi(\bar{a}', \bar{c}')$ for some/any $\bar{a}'$ such that $\models \varphi(\bar{a}', \bar{b}') \wedge \psi(\bar{a}', \bar{c}')$. So we have shown that the complement of E is $\bigvee$-definable over $\emptyset$.)
(ii) By [27, Theorem 4.15, Corollary 1.5] and Remark 3.2, without loss of generality E lives on finite tuples. It is well-known that E is equivalent to a conjunction of equivalence relations each of which is defined by a countable collection of formulas over $\emptyset$ and is also bounded (see [31, Lemma 3.1.3]). So we may assume that E is defined by a countable collection of formulas. Then $\mathfrak{C}/E$ is a compact space, metrizable via an $\mathrm{Aut}(\mathfrak{C})$-invariant metric d (see [22, Section 3, p. 237]). Define $d_E(\bar{x}, \bar{y}) := d(\bar{x}/E, \bar{y}/E)$. This is clearly a CL-formula, and we see that each $\bar{a}/E$ is interdefinable with the code of $d_E(\bar{x}, \bar{a})$. □

Let $\mathrm{acl}^{eq}_{CL}(\emptyset)$ denote the collection of CL-imaginaries which have a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. Likewise $\mathrm{bdd}^{heq}(\emptyset)$ is the collection of hyperimaginaries with a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. By Lemma 3.3 and Remark 3.2, we get

Corollary 3.4. *(i) Up to interdefinability,* $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *coincides with* $\mathrm{bdd}^{heq}(\emptyset)$.
(ii) Moreover, $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *is interdefinable with the collection of finitary CL-imaginaries with a bounded number of conjugates under* $\mathrm{Aut}(\bar{M})$.

We now appeal to the local stability results in [3] (which go somewhat beyond what we deduced purely from Grothendieck in Section 3 of [13]). Fix a finite tuple $\bar{x}$ of variables and consider $\Delta_{\mathrm{st}}(\bar{x})$, the collection of all stable formulas (without parameters) $\phi(\bar{x}, \bar{y})$ of T_{CL}, where $\bar{y}$ varies and where stability of $\phi(\bar{x}, \bar{y})$ means that for all $\epsilon > 0$ there do not exist $\bar{a}_i, \bar{b}_i$ for $i < \omega$ (in the monster model) such that for all $i < j$, $|\phi(\bar{a}_i, \bar{b}_j) - \phi(\bar{a}_j, \bar{b}_i)| \geq \epsilon$. (By Ramsey theorem and compactness, $\phi(\bar{x}, \bar{y})$ is stable if and only if whenever $(\bar{a}_i, \bar{b}_i)_{i<\omega}$ is indiscernible, then $\phi(\bar{a}_i, \bar{b}_j) = \phi(\bar{a}_j, \bar{b}_i)$ for $i < j$.) For an n-tuple $\bar{b}$ and set A of parameters (including possibly CL-imaginaries), $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/A)$ is the function taking the formula $\phi(\bar{x}, \bar{a})$ to $\phi(\bar{b}, \bar{a})$, where $\phi(\bar{x}, \bar{y}) \in \Delta_{\mathrm{st}}$ and $\phi(\bar{x}, \bar{a})$ is over A (i.e invariant under $\mathrm{Aut}(\bar{M}/A)$). By definition, a *complete* Δ_{st}*-type over* A is something of the form $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/A)$ (and $\bar{b}$ is a realization of it).

Remark 3.5. For any $\bar{b}$, $\mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(\emptyset))$ (in the classical case) coincides with $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset))$ in the continuous framework, meaning that $\mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(\emptyset)) = \mathrm{tp}(\bar{b}'/\mathrm{bdd}^{heq}(\emptyset))$ if and only if $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset)) = \mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}'/\mathrm{acl}^{eq}_{CL}(\emptyset))$.

Proof. Using Corollary 3.4, the left hand side always implies the right hand side. For the other direction, since $\bar{x} \equiv_{\mathrm{bdd}^{heq}(\emptyset)} \bar{y}$ is a bounded, type-definable over $\emptyset$ equivalence relation (in fact, it is exactly E_{KP}), it is enough to show that for any bounded, type-definable over $\emptyset$ equivalence relation E, whenever $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/\operatorname{acl}^{eq}_{CL}(\emptyset)) = \mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}'/\operatorname{acl}^{eq}_{CL}(\emptyset))$, then $E(\bar{b}, \bar{b}')$. Without loss of generality we may assume that E is defined by countably many formulas. Let $d_E(\bar{x}, \bar{y})$ be the CL-formula from the proof of Lemma 3.3(ii). As E is bounded, $d_E(\bar{x}, \bar{y})$ is stable (because for every indiscernible sequence $(\bar{a}_i, \bar{b}_i)_{i<\omega}$ all $\bar{a}_i$'s are in a single E-class and all $\bar{b}_i$'s are in a single E-class, and so $d_E(\bar{a}_i, \bar{b}_j) = d(\bar{a}_i/E, \bar{b}_j/E)$ is constant for all $i, j < \omega$). The code of $d_E(\bar{x}, \bar{b})$ is interdefinable with $\bar{b}/E$, hence it is in $\operatorname{acl}^{eq}_{CL}(\emptyset)$, and so $d_E(\bar{x}, \bar{b})$ is over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. Since clearly $d_E(\bar{b}, \bar{b}) = 0$, we conclude that $d_E(\bar{b}', \bar{b}) = 0$ which means that $E(\bar{b}, \bar{b}')$. □

If M is a model, then $p = \mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/M)$ can be identified with the collection of functions $f_\phi \colon M^n \to \mathbb{R}$ taking $\bar{a} \in M^n$ to $\phi(\bar{b}, \bar{a})$, for $\phi(\bar{x}, \bar{y}) \in \Delta_{\mathrm{st}}$. The type $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/M)$ is said to be *definable* (over M) if the functions f_ϕ are induced by CL-formulas over M; it is *definable over* A if the f_ϕ's are induced by CL-formulas over A. A $\varphi(\bar{x}, \bar{y})$*-definition of* p is a CL-formula $\chi(\bar{y})$ such that $\varphi(\bar{b}, \bar{a}) = \chi(\bar{a})$ for all $\bar{a}$ from M.

The following is a consequence of the local theory developed in Section 7 of [3] and the discussion around gluing in Section 8 of the same paper (see the proof of [3, Proposition 8.7]). We restrict ourselves to the case needed, i.e. over $\emptyset$.

Fact 3.6. *Let $p(\bar{x})$ be a (CL-)complete Δ_{st}-type over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. Then for any model M (which note contains $\operatorname{acl}^{eq}_{CL}(\emptyset)$) there is a unique complete Δ_{st}-type $q(\bar{x})$ over M such that $q(\bar{x})$ extends $p(\bar{x})$ and q is definable over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. We say $q = p|M$. In particular, if $M \prec N$, then $p|M$ is precisely the restriction of $p|N$ to M.*

Definition 3.7. We say that $\bar{b}$ is *stably independent* from B (or that $\bar{b}$ and B are *stably independent*) if $\mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/B)$ equals the restriction of $p|M$ to B, where M is some model containing B and $p = \mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{b}/\operatorname{acl}^{eq}_{CL}(\emptyset))$.

Stable independence is clearly invariant under automorphisms. The usual Erdös-Rado arguments, together with Fact 3.6 give:

Corollary 3.8. *Let q be a complete Δ_{st}-type over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. Then there is an infinite sequence $(\bar{b}_i : i < \omega)$ of realizations of q which is indiscernible and such that $\bar{b}_i$ is stably independent from $\{\bar{b}_j : j < i\}$ for all i.*

The following consequence of Fact 3.6 will also be important for us.

Corollary 3.9. *Suppose we have finite tuples $\bar{a}$ and $\bar{b}$ from the (classical) model $\mathfrak{C}$. Suppose that $\bar{a}$ is stably independent from $\bar{b}$. Then for any stable CL-formula $\psi(\bar{x}, \bar{y})$ (over $\emptyset$), the value of $\psi(\bar{a}, \bar{b})$ depends only on $\mathrm{tp}(\bar{a}/\operatorname{bdd}^{heq}(\emptyset))$ and $\mathrm{tp}(\bar{b}/\operatorname{bdd}^{heq}(\emptyset))$ (in the sense of the classical structure $\mathfrak{C}$).*

Proof. Let $p(\bar{x}) := \mathrm{tp}_{\Delta_{\mathrm{st}}}(\bar{a}/\operatorname{acl}^{eq}_{CL}(\emptyset))$. The $\psi(\bar{x}, \bar{y})$-type of $p|\mathfrak{C}$ is by Fact 3.6 definable by a CL-formula $\chi(\bar{y})$ over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. So assuming the stable independence of $\bar{a}$ and $\bar{b}$, by definition and Fact 3.6, the value of $\psi(\bar{a}, \bar{b})$ is equal to $\chi(\bar{b})$, which by Remark 3.5 depends only on $\operatorname{tp}(\bar{b}/\operatorname{bdd}^{heq}(\emptyset))$. If $\bar{a}$ is replaced by another realization $\bar{a}'$ of p (equivalently, $\bar{a}' \models \operatorname{tp}(\bar{a}/\operatorname{bdd}^{heq}(\emptyset))$) which is stably independent from another realization $\bar{b}'$ of $\operatorname{tp}(\bar{b}/\operatorname{bdd}^{heq}(\emptyset))$, then the above shows that $\psi(\bar{a}', \bar{b}') = \chi(\bar{b}') = \chi(\bar{b}) = \psi(\bar{a}, \bar{b})$. □

Proposition 3.10. *Let $\mu = \mu_{\bar{x}}$ be a global, $\emptyset$-definable Keisler measure. Let $\bar{a}$ and $\bar{b}$ be tuples of the same length from $\mathfrak{C}$, with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, and stably independent. Let $p(\bar{x}, \bar{a})$ be a complete type over $\bar{a}$ which is "μ-wide" in the sense that every formula in $p(\bar{x}, \bar{a})$ gets μ-measure > 0. Then the partial type $p(\bar{x}, \bar{a}) \cup p(\bar{x}, \bar{b})$ is also μ-wide (again in the sense that every formula implied by it has μ-measure > 0).*

Proof. By definition, we have to show that if $\phi(\bar{x}, \bar{a})$ is a formula with μ-measure > 0, then $\phi(\bar{x}, \bar{a}) \wedge \phi(\bar{x}, \bar{b})$ has μ-measure > 0. By $\emptyset$-definability of μ, the function $\psi(\bar{y}, \bar{z})$ defined to be $\mu(\phi(\bar{x}, \bar{y}) \wedge \phi(\bar{x}, \bar{z}))$ is definable over $\emptyset$, i.e. is a CL-formula without parameters. Moreover, by Proposition 2.25 of [12], $\psi(\bar{y}, \bar{z})$ is stable. Bearing in mind Remark 3.5, let, by Corollary 3.8, $(\bar{a}_i : i < \omega)$ be an indiscernible sequence of realizations of $q := \operatorname{tp}(\bar{a}/\operatorname{bdd}^{heq}(\emptyset))$ such that $\bar{a}_j$ and $\bar{a}_i$ are stably independent for all $i < j$ (equivalently for some $i < j$). Since μ is $\operatorname{Aut}(\mathfrak{C})$-invariant, we see that $\mu(\phi(\bar{x}, \bar{a}_i))$ is positive and constant for all i, and $\mu(\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j))$ is positive (and constant) for $i \neq j$. In particular, $\psi(\bar{a}_0, \bar{a}_1) > 0$. By Corollary 3.9, $\psi(\bar{a}, \bar{b}) > 0$, which is what we had to prove. □

Proposition 3.11. *Suppose that (the classical, first order theory) T is definably amenable. Then T is G-compact. In fact, the diameter of each Lascar strong type (over $\emptyset$) is bounded by 2.*

Proof. We have to show that if $\bar{a}$, $\bar{b}$ are tuples of the same (but possibly infinite) length and with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, then they have the same Lascar strong type; more precisely, the Lascar distance between them is ≤ 2.

Observe that, by compactness, without loss of generality, we can and do assume that $\bar{a}$ and $\bar{b}$ are finite tuples (this is because if we show that all corresponding finite subtuples of $\bar{a}$ and $\bar{b}$ are at Lascar distance at most 2, then so are $\bar{a}$ and $\bar{b}$).

Assume first that $\bar{a}$ and $\bar{b}$ are stably independent in the sense of Definition 3.7. Fix a model M_0 and enumerate it. We will find a copy M of M_0 such that $\operatorname{tp}(\bar{a}/M) = \operatorname{tp}(\bar{b}/M)$ (which immediately yields that $\bar{a}$ and $\bar{b}$ have the same Lascar strong type; in fact, $d_L(\bar{a}, \bar{b}) \leq 1$). By compactness, given a consistent formula $\phi(\bar{y})$ in finitely many variables, it suffices to find some realization $\bar{m}$ of $\phi(\bar{y})$ such that $\operatorname{tp}(\bar{a}/\bar{m}) = \operatorname{tp}(\bar{b}/\bar{m})$. By assumption, let $\mu_{\bar{y}}$ be a $\emptyset$-definable, global Keisler measure concentrating on $\phi(\bar{y})$. Let $p(\bar{y}, \bar{a})$ be a complete type over $\bar{a}$ which is μ-wide. By Proposition 3.10, $p(\bar{y}, \bar{a}) \cup p(\bar{y}, \bar{b})$ is also μ-wide, in particular consistent. So let $\bar{m}$ realize it.

In general, given finite tuples $\bar{a}$, $\bar{b}$ with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, let $\bar{d}$ have the same type over $\operatorname{bdd}^{heq}(\emptyset)$ and be stably independent from $\{\bar{a}, \bar{b}\}$ (by Remark

3.5 and Fact 3.6). By what we have just shown, $d_L(\bar{a},\bar{d}) \leq 1$ and $d_L(\bar{b},\bar{d}) \leq 1$. So $d_L(\bar{a},\bar{b}) \leq 2$. □

4. Amenability implies G-compactness: the general case

Let T be an arbitrary theory, $\mathfrak{C} \models T$ a monster model, and $\bar{c}$ an enumeration of $\mathfrak{C}$. The goal of this section is to prove Theorem 0.1; in fact, we will get more precise information:

Theorem 4.1. *If T is amenable, then T is G-compact. In fact, the diameter of each Lascar strong type (over $\emptyset$) is bounded by 4.*

Before we start our analysis towards the proof of Theorem 4.1, let us first note the analogous statement for extreme amenability, which is much easier to prove.

Proposition 4.2. *If $p(\bar{x}) \in S(\emptyset)$ is extremely amenable, then $p(\bar{x})$ is a single Lascar strong type. Moreover, the Lascar diameter of $p(\bar{x})$ is at most 2.*

In particular, if T is extremely amenable, then the Lascar strong types coincide with complete types (over $\emptyset$), i.e. the Lascar Galois group $\mathrm{Gal}_L(T)$ is trivial. Furthermore, if T is extremely amenable, the Lascar distance between any two elements which have the same type is at most 1.

Proof. Choose $\mathfrak{C}$ so that $\bar{x}$ is short in $\mathfrak{C}$. Let $q \in S_p(\mathfrak{C})$ be invariant under $\mathrm{Aut}(\mathfrak{C})$. Fix $\bar{\alpha} \models q$ (in a bigger model). Take a small $M \prec \mathfrak{C}$ and choose $\bar{\beta} \in \mathfrak{C}$ such that $\bar{\beta} \models q|_M$. Then $\bar{\alpha}\, E_L\, \bar{\beta}$. But also, for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, $\sigma(\bar{\beta}) \models \sigma(q)|_{\sigma[M]} = q|_{\sigma[M]}$, and so $\sigma(\bar{\beta})\, E_L\, \bar{\alpha}$. Therefore, $\sigma(\bar{\beta})\, E_L\, \bar{\beta}$ for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, which shows that $p(\bar{x})$ is a single Lascar strong type.

For the "moreover part" notice that, in the above argument, both $d_L(\bar{\alpha},\bar{\beta})$ and $d_L(\sigma(\bar{\beta}),\bar{\alpha})$ are bounded by 1.

The "in particular part" follows immediately from the first part, but we also give a shorter proof suggested by the referee which yields additionally Lascar distance at most 1. Consider any $\bar{a} \equiv \bar{b}$. By assumption, there exists a model M enumerated as $\bar{m}$ such that $\mathrm{tp}(\bar{m}/\mathfrak{C})$ is invariant. Then $\bar{a} \equiv_M \bar{b}$, so $d_L(\bar{a},\bar{b}) \leq 1$. □

Recall from Definition 2.14 that by a *relatively type-definable subset* of $\mathrm{Aut}(\mathfrak{C})$ we mean a subset of the form

$$A_{\pi,\bar{a},\bar{b}} := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}),\bar{b}))\}$$

for some partial type $\pi(\bar{x},\bar{y})$ (without parameters), where $\bar{x}$ and $\bar{y}$ are short tuples of variables and $\bar{a}$, $\bar{b}$ are from $\mathfrak{C}$. Without loss $\bar{x}$ is of the same length as $\bar{y}$ and $\bar{a} = \bar{b}$, and then we write $A_{\pi,\bar{a}}$. In fact, the following remark is very easy.

Remark 4.3. For any partial types $\pi_1(\bar{x}_1,\bar{y}_1)$ and $\pi_2(\bar{x}_2,\bar{y}_2)$ and tuples $\bar{a}_1,\bar{a}_2,\bar{b}_1,\bar{b}_2$ in $\mathfrak{C}$ corresponding to $\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2$, one can find partial types $\pi_1'(\bar{x},\bar{y})$ and $\pi_2'(\bar{x},\bar{y})$ with $\bar{x}$ of the same length (by which we also mean of the same sorts) as $\bar{y}$ and a tuple $\bar{a}$ in $\mathfrak{C}$ corresponding to $\bar{x}$ such that $A_{\pi_1,\bar{a}_1,\bar{b}_1} = A_{\pi_1',\bar{a}}$ and $A_{\pi_2,\bar{a}_2,\bar{b}_2} = A_{\pi_2',\bar{a}}$.

For a short tuple $\bar{\alpha}$ and a short tuple of parameters $\bar{b}$, a subset of $\mathrm{Aut}(\mathfrak{C})$ is called *relatively $\bar{\alpha}$-type-definable over $\bar{b}$* if it is of the form $A_{\pi,\bar{\alpha},\bar{b}}$ for some partial type $\pi(\bar{x},\bar{y})$.

The next fact was observed in [23].

Fact 4.4 (Proposition 5.2 of [23])**.** *If G is a closed, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *(with* $\mathrm{Aut}(\mathfrak{C})$ *equipped with the pointwise convergence topology), then* $\mathrm{Autf}_L(\mathfrak{C}) \leq G$.

Using an argument similar to the proof of Fact 4.4, we will first show

Proposition 4.5. *If G is a relatively type-definable, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$*, then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G$.

Proof. Let σ_i, $i < \lambda$, be a set of representatives of the left cosets of G in $\mathrm{Aut}(\mathfrak{C})$ (so λ is bounded). Then

$$G' := \bigcap_{\sigma \in \mathrm{Aut}(\mathfrak{C})} G^\sigma = \bigcap_{i<\lambda} G^{\sigma_i}$$

is a normal, bounded index subgroup of $\mathrm{Aut}(\mathfrak{C})$ (where $G^\sigma := \sigma G \sigma^{-1}$).

Let us show now that G' is relatively type-definable. We have $G = A_{\pi,\bar{a}} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}),\bar{a})\}$ for some type $\pi(\bar{x},\bar{y})$ (with short $\bar{x}$, $\bar{y}$) and tuple $\bar{a}$ in $\mathfrak{C}$. Then $G^{\sigma_i} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\sigma_i(\bar{a})),\sigma_i(\bar{a}))\}$, so putting $\bar{a}' = \langle \sigma_i(\bar{a})\rangle_{i<\lambda}$, $\bar{x}' = \langle \bar{x}_i\rangle_{i<\lambda}$, $\bar{y}' = \langle \bar{y}_i\rangle_{i<\lambda}$ (where $\bar{x}_i$ and $\bar{y}_i$ are copies of $\bar{x}$ and $\bar{y}$, respectively) and $\pi'(\bar{x}',\bar{y}') = \bigcup_{i<\lambda} \pi(\bar{x}_i,\bar{y}_i)$ (as a set of formulas), we see that

$$(*) \qquad G' = A_{\pi',\bar{a}'} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi'(\sigma(\bar{a}'),\bar{a}')\},$$

which is clearly relatively type-definable.

The orbit equivalence relation E of the action of G' on the set of realizations of $\mathrm{tp}(\bar{a}'/\emptyset)$ is a bounded equivalence relation. This relation is type-definable, because

$$\bar{\alpha}\, E\, \bar{\beta} \iff (\exists g \in G')(g(\bar{\alpha}) = \bar{\beta}) \iff (\exists \bar{b}')(\pi'(\bar{b}',\bar{a}') \wedge \bar{a}'\bar{\alpha} \equiv \bar{b}'\bar{\beta}).$$

But E is also invariant (as G' is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$), so E is type-definable over $\emptyset$. Therefore, E is refined by E_{KP}.

Now, take any $\sigma \in \mathrm{Autf}_{KP}(\mathfrak{C})$. By the last conclusion, there is $\tau \in G'$ such that $\sigma(\bar{a}') = \tau(\bar{a}')$. Then $\tau^{-1}\sigma(\bar{a}') = \bar{a}'$ and $\sigma = \tau(\tau^{-1}\sigma)$. Since $(*)$ implies that $G' \cdot \mathrm{Fix}(\bar{a}') = G'$, we get $\sigma \in G'$. Thus, $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G' \leq G$. □

Recall that a subset C of a group is called *(left) generic* if finitely many left translates of it covers the whole group; C is called *symmetric* if it contains the neutral element and $C^{-1} = C$.

Corollary 4.6. *If $\{C_i : i \in \omega\}$ is a family of relatively definable, generic, symmetric subsets of* $\mathrm{Aut}(\mathfrak{C})$ *such that $C_{i+1}^2 \subseteq C_i$ for all $i \in \omega$, then $\bigcap_{i\in\omega} C_i$ is a subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *containing* $\mathrm{Autf}_{KP}(\mathfrak{C})$.

Proof. It is clear that $\bigcap_{i\in\omega} C_i$ is a subgroup of $\mathrm{Aut}(\mathfrak{C})$, and it is easy to show that it has bounded index (at most $2^{\aleph_0}$). Moreover, it is clearly relatively type-definable. Thus, the fact that it contains $\mathrm{Autf}_{KP}(\mathfrak{C})$ follows from Proposition 4.5. □

Lemma 4.7. *i) Let $\pi(\bar{x},\bar{y})$ be a partial type (over $\emptyset$) and $\bar{a}$, $\bar{b}$ short tuples from $\mathfrak{C}$ corresponding to $\bar{x}$ and $\bar{y}$, respectively. Then $A^{-1}_{\pi,\bar{a},\bar{b}} = A_{\pi',\bar{b},\bar{a}}$, where $\pi'(\bar{y},\bar{x}) := \pi(\bar{x},\bar{y})$ (i.e. the type π with the exchanged roles of variables).*
ii) Let $n \geq 2$ be a natural number. Let $\bar{x}$, $\bar{y}$ and $\bar{x}_1,\dots,\bar{x}_n$ be disjoint, short tuples of variables of the same length. Then there exists a partial type $\Phi_n(\bar{x},\bar{y},\bar{x}_1,\dots,\bar{x}_n)$ such that for every partial types $\pi_1(\bar{x}_1,\bar{y}),\dots,\pi_n(\bar{x}_n,\bar{y})$ and tuple $\bar{a}$ corresponding to $\bar{x}$ one has

$$A_{\pi_1,\bar{a}}\cdot \ldots \cdot A_{\pi_n,\bar{a}} = A_{\pi,\bar{a}},$$

where

$$\pi(\bar{x},\bar{y}) := (\exists \bar{x}_1,\dots,\bar{x}_n)(\pi_1(\bar{x}_1,\bar{y})\wedge\dots\wedge\pi_n(\bar{x}_n,\bar{y})\wedge\Phi_n(\bar{x},\bar{y},\bar{x}_1,\dots,\bar{x}_n)).$$

Proof. (i) follows immediately from the fact that for any $\sigma\in\mathrm{Aut}(\mathfrak{C})$

$$\mathfrak{C}\models\pi(\sigma(\bar{a}),\bar{b}) \iff \mathfrak{C}\models\pi(\bar{a},\sigma^{-1}(\bar{b})) \iff \mathfrak{C}\models\pi'(\sigma^{-1}(\bar{b}),\bar{a}).$$

(ii) We will show that for $n=2$ the type $\Phi_2(\bar{x},\bar{y},\bar{x}_1,\bar{x}_2) := (\bar{x}\bar{x}_1 \equiv \bar{x}_2\bar{y})$ and for $n\geq 3$ the type $\Phi_n(\bar{x},\bar{y},\bar{x}_1,\dots,\bar{x}_n)$ defined as

$$(\exists \bar{z}_1,\dots,\bar{z}_{n-2})(\bar{x}\bar{z}_{n-2}\equiv\bar{x}_n\bar{y}\wedge\bar{z}_{n-2}\bar{z}_{n-3}\equiv\bar{x}_{n-1}\bar{y}\wedge\dots\wedge\bar{z}_2\bar{z}_1\equiv\bar{x}_3\bar{y}\wedge\bar{z}_1\bar{x}_1\equiv\bar{x}_2\bar{y})$$

is as required.

First, let us see that $A_{\pi_1,\bar{a}}\cdot\ldots\cdot A_{\pi_n,\bar{a}}\subseteq A_{\pi,\bar{a}}$. Take σ from the left hand side, i.e. $\sigma=\sigma_1\ldots\sigma_n$, where $\models\pi_i(\sigma_i(\bar{a}),\bar{a})$. Then $\models\pi(\sigma(\bar{a}),\bar{a})$ is witnessed by $\bar{x}_i:=\sigma_i(\bar{a})$ for $i=1,\dots,n$ and $\bar{z}_i:=(\sigma_1\ldots\sigma_{i+1})(\bar{a})$ for $i=1,\dots,n-2$. So $\sigma\in A_{\pi,\bar{a}}$.

Finally, we will justify that $A_{\pi_1,\bar{a}}\cdot\ldots\cdot A_{\pi_n,\bar{a}}\supseteq A_{\pi,\bar{a}}$. Consider the case $n\geq 3$. Take any σ such that $\models\pi(\sigma(\bar{a}),\bar{a})$. Let $\bar{a}_1,\dots,\bar{a}_n$ be witnesses for $\bar{x}_1,\dots,\bar{x}_n$, and $\bar{b}_1,\dots,\bar{b}_{n-2}$ be witnesses for $\bar{z}_1,\dots,\bar{z}_{n-2}$, i.e.:

(1) $\models\pi_i(\bar{a}_i,\bar{a})$ for $i=1,\dots,n$, and
(2) $\sigma(\bar{a})\bar{b}_{n-2}\equiv\bar{a}_n\bar{a}\wedge\bar{b}_{n-2}\bar{b}_{n-3}\equiv\bar{a}_{n-1}\bar{a}\wedge\dots\wedge\bar{b}_2\bar{b}_1\equiv\bar{a}_3\bar{a}\wedge\bar{b}_1\bar{a}_1\equiv\bar{a}_2\bar{a}$.

By (2), there are $\tau_1,\dots,\tau_{n-1}\in\mathrm{Aut}(\mathfrak{C})$ mapping the right hand sides of the equivalences in (2) to the left hand sides. Then $\tau_1(\bar{a}_n)=\sigma(\bar{a})$, so $\tau_1^{-1}\sigma(\bar{a})=\bar{a}_n$, so $\tau_1^{-1}\sigma\in A_{\pi_n,\bar{a}}$ by (1). Next, $\tau_1(\bar{a})=\bar{b}_{n-2}=\tau_2(\bar{a}_{n-1})$, so $\tau_2^{-1}\tau_1(\bar{a})=\bar{a}_{n-1}$, so $\tau_2^{-1}\tau_1\in A_{\pi_{n-1},\bar{a}}$ by (1). We continue in this way, obtaining in the last step: $\tau_{n-1}(\bar{a})=\bar{a}_1$, so $\tau_{n-1}\in A_{\pi_1,\bar{a}}$ by (1). Therefore,

$$\sigma=\tau_{n-1}(\tau_{n-1}^{-1}\tau_{n-2})\ldots(\tau_2^{-1}\tau_1)(\tau_1^{-1}\sigma)\in A_{\pi_1,\bar{a}}\cdot\ldots\cdot A_{\pi_n,\bar{a}}.$$

For $n=2$, in (2), we just have $\sigma(\bar{a})\bar{a}_1\equiv\bar{a}_2\bar{a}$, so taking $\tau_1\in\mathrm{Aut}(\mathfrak{C})$ which maps $\bar{a}_2\bar{a}$ to $\sigma(\bar{a})\bar{a}_1$, we get $\tau_1^{-1}\sigma\in A_{\pi_2,\bar{a}}$ and $\tau_1\in A_{\pi_1,\bar{a}}$, hence $\sigma\in A_{\pi_1,\bar{a}}\cdot A_{\pi_2,\bar{a}}$. □

Corollary 4.8. *Let $\pi_1(\bar{x},\bar{y}),\dots,\pi_n(\bar{x},\bar{y})$ be partial types, $\bar{a}$ a tuple corresponding to $\bar{x}$ and $\bar{y}$, and $\epsilon_1,\dots,\epsilon_n \in \{-1,1\}$.*
(i) Then

$$A_{\pi_1,\bar{a}}^{\epsilon_1}\cdot\ldots\cdot A_{\pi_n,\bar{a}}^{\epsilon_n} = \bigcap\left\{A_{\varphi_1,\bar{a}}^{\epsilon_1}\cdot\ldots\cdot A_{\varphi_n,\bar{a}}^{\epsilon_n} : \pi_1 \vdash \varphi_1,\dots,\pi_n \vdash \varphi_n\right\}.$$

ii) If $A_{\pi_1,\bar{a}}^{\epsilon_1}\cdot\ldots\cdot A_{\pi_n,\bar{a}}^{\epsilon_n}$ is contained in a relatively definable subset A of $\mathrm{Aut}(\mathfrak{C})$, then there are $\varphi_i(\bar{x},\bar{y})$ implied by $\pi_i(\bar{x},\bar{y})$ for $i=1,\dots,n$, such that $A_{\varphi_1,\bar{a}}^{\epsilon_1}\cdot\ldots\cdot A_{\varphi_n,\bar{a}}^{\epsilon_n} \subseteq A$.

Proof. This follows from Lemma 4.7, using compactness and the fact that $\mathfrak{C}$ is a monster model. But let us give some details.

By Lemma 4.7(i), we can clearly assume that $\epsilon_i = 1$ for all i. Then item (i) follows directly from Lemma 4.7(ii). So it remains to show item (ii).

Take a formula $\psi(\bar{x}',\bar{y}')$ (where $\bar{x}'$ and $\bar{y}'$ are of the same length and are disjoint from both $\bar{x}$ and $\bar{y}$) and $\bar{a}'$ such that $A = A_{\psi,\bar{a}'}$. We can also treat ψ as $\psi(\bar{x}\bar{x}',\bar{y}\bar{y}')$, and then $A = A_{\psi,\bar{a}\bar{a}'}$. Similarly, π_i can be treated as $\pi_i(\bar{x}\bar{x}',\bar{y}\bar{y}')$, and then the original product $A_{\pi_1,\bar{a}}\cdot\ldots\cdot A_{\pi_n,\bar{a}}$ can be written as $A_{\pi_1,\bar{a}\bar{a}'}\cdot\ldots\cdot A_{\pi_n,\bar{a}\bar{a}'}$. By Lemma 4.7(ii) and strong κ-homogeneity of $\mathfrak{C}$, we get that the type

$$(\exists \bar{x}_1\bar{x}_1',\dots,\bar{x}_n\bar{x}_n')(\pi_1(\bar{x}_1 x_1',\bar{a}\bar{a}')\wedge\dots\wedge\pi_n(\bar{x}_n\bar{x}_n',\bar{a}\bar{a}')\wedge\Phi_n(\bar{x}\bar{x}',\bar{a}\bar{a}',\bar{x}_1\bar{x}_1',\dots,\bar{x}_n\bar{x}_n'))$$

in conjunction with $\bar{x}\bar{x}' \equiv \bar{a}\bar{a}'$ implies the type $\psi(\bar{x}\bar{x}',\bar{a}\bar{a}')$. Hence, by compactness, each type $\pi_i(\bar{x}\bar{x}',\bar{y}\bar{y}')$ can be replaced by a formula $\varphi_i(\bar{x}\bar{x}',\bar{y}\bar{y}')$ implied by $\pi_i(\bar{x}\bar{x}',\bar{y}\bar{y}')$ so that the above implication is still valid. Since the types π_i use only variables $\bar{x},\bar{y}$, the formulas φ_i can also be chosen only in variables $\bar{x},\bar{y}$. Then, by the above implication (with the π_i's replaced by φ_i's) and Lemma 4.7(ii), we get that $A_{\varphi_1,\bar{a}}\cdot\ldots\cdot A_{\varphi_n,\bar{a}} = A_{\varphi_1,\bar{a}\bar{a}'}\cdot\ldots\cdot A_{\varphi_n,\bar{a}\bar{a}'} \subseteq A$. □

Lemma 4.9. *Let $p(\bar{x}) \in S(\emptyset)$ with $\bar{x}$ short, $q \in S_{\bar{p}}(\mathfrak{C})$, $M \prec \mathfrak{C}$ small, and $\bar{\alpha} \models q|_M$. Then $A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}\bar{\alpha} \subseteq \{\bar{\beta} \subset \mathfrak{C} : d_L(\bar{\alpha},\bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}$.*

Proof. Let us start from the following

Claim 1: For any $\bar{\beta} \models q|_{\bar{\alpha}}$, $d_L(\bar{\beta},\bar{\alpha}) \leq 1$.

Proof. Take $\bar{\gamma} \models q|_{M\bar{\alpha}}$. Then $d_L(\bar{\gamma},\bar{\alpha}) \leq 1$, so the conclusion follows from the fact that $\bar{\beta} \equiv_{\bar{\alpha}} \bar{\gamma}$. □(claim)

Now, consider any $\sigma_1,\sigma_2,\sigma_3,\sigma_4 \in A_{q|_{\bar{\alpha}},\bar{\alpha}}$. Then $\sigma_i(\bar{\alpha}) \models q|_{\bar{\alpha}}$, so, by the claim, we get $d_L(\sigma_i(\bar{\alpha}),\bar{\alpha}) \leq 1$. Therefore, $d_L(\sigma_4^{-1}(\bar{\alpha}),\bar{\alpha}) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\sigma_3^{-1}(\bar{\alpha})) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\bar{\alpha}) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\sigma_2(\bar{\alpha})) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\bar{\alpha}) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\sigma_1(\bar{\alpha})) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}),\bar{\alpha}) \leq 4$. □

The proof of the next lemma uses a version of the stabilizer theorem obtained in [13, Corollary 2.12]. We will not recall here all the terminology involved in [13, Corollary 2.12]; the reader may consult Subsections 2.1, 2.2, and 2.3 of [13]. Let us recall here the main things. $\mathcal{L}_k^{\mathrm{gen}}$, $k \in \omega$, is a recursively defined notion of largeness of $\bigvee$-definable sets concentrated on a definable group G, which is invariant under left translations. Passing to a sufficiently saturated extension $\bar{G}$ of G, for any $\bigvee$-definable subset Y, $\mathcal{L}_0^{\text{gen}}(Y)$ means that $Y \neq \emptyset$ and $\mathcal{L}_k^{\text{gen}}(Y)$ means precisely that $\{g \in \bar{G} : \mathcal{L}_{k-1}^{\text{gen}}(gY \cap Y)\}$ is generic (i.e. finitely many left translates of it cover $\bar{G}$). Next, $\text{St}_{\mathcal{L}_k}(Y) := \{g : \mathcal{L}_k(gY \cap Y)\}$ is an operator from the class of $\bigvee$-definable sets concentrated on G to itself (see the paragraphs after Remark 2.3 in [13]). We would like to emphasize that $\text{St}_{\mathcal{L}_k}(Y)$ need not be a subgroup of G. (In [13], we are more precise and consider the class of $\bigvee$-positively definable sets, which is essential in the applications in [13], but here we do not care about positive definability). We will need the following basic remark.

Remark 4.10. If a $\bigvee$-definable set Y is invariant under left translations by the elements of some subgroup H of G, then $\text{St}_{\mathcal{L}_k}(Y)$ is invariant under both left and right translations by the elements of H.

Instead of stating [13, Corollary 2.12] in full generality and with the full power, we give a particular case, which is sufficient for our application.

Fact 4.11. *Let G be a group, and $A \subseteq G$. Let $\mathcal{A}$ be a Boolean algebra of subsets of G which is invariant under left translations and includes A and all sets of the form $(g_1A \cap \dots \cap g_kA)A$ (with $k \geq 1$ and $g_1, \dots, g_k \in G$). Let μ be a left-invariant, finitely additive measure on $\mathcal{A}$ with $\mu(A) > 0$. Then there exist $l \in \mathbb{N}_{>0}$ and $g_1, \dots, g_n \in G$ such that for $A' := A \cap g_1A \cap \dots \cap g_nA$, the set $S := \text{St}_{\mathcal{L}_{l-1}^{gen}}(A')$ (computed with respect to $\text{Th}(G, \cdot, A)$) is generic, symmetric, and satisfies $S^{16} \subseteq AAA^{-1}A^{-1}$.*

Proof. Apply [13, Corollary 2.12] for $\mathcal{B} := \{A\}$, $N := 16$, $\mathcal{D} := \mathcal{A}$, and $m := \mu$. As a result we obtain $l \in \mathbb{N}_{>0}$ and $g_1, \dots, g_n \in G$ such that for $A' := A \cap g_1A \cap \dots \cap g_nA$, $S := \text{St}_{\mathcal{L}_{l-1}^{\text{gen}}}(A')$ is generic as a $\bigvee$-definable set in $(G, \cdot, A)$, symmetric, and satisfies $S^{16} \subseteq AAA^{-1}A^{-1}$. In particular, S is a generic subset of G, i.e. finitely many left translates of S cover G. □

Lemma 4.12. *Assume* $\text{Aut}(\mathfrak{C})$ *is relatively amenable which is witnessed by an* $\text{Aut}(\mathfrak{C})$*-invariant, regular, Borel probability measure $\tilde{\mu}$ on $S_{\bar{c}}(\mathfrak{C})$. Let μ be the induced* $\text{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure on the Boolean algebra $\mathcal{A}$ generated by relatively type-definable subsets of* $\text{Aut}(\mathfrak{C})$*, as described in Corollary 2.15. Suppose $A \subseteq \text{Aut}(\mathfrak{C})$ is relatively type-definable with $\mu(A) > 0$ and $AAA^{-1}A^{-1} \subseteq A'$ for some relatively definable $A' \subseteq \text{Aut}(\mathfrak{C})$. Then there exists a relatively type-definable, generic, symmetric $Y \subseteq \text{Aut}(\mathfrak{C})$ such that $Y^8 \subseteq A'$.*

Proof. By Lemma 4.7, relatively type-definable sets are closed under taking products and inversions, and one can easily check that also under left translations.

Claim 1: There exists a generic and symmetric set $S \subseteq \text{Aut}(\mathfrak{C})$ such that:

(1) $S^{16} \subseteq AAA^{-1}A^{-1}$,
(2) $S = \{\sigma \in \text{Aut}(\mathfrak{C}) : \text{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$, where $\bar{a}$ is a short tuple (more precisely, a tuple of finitely many conjugates by elements of $\text{Aut}(\mathfrak{C})$ of the tuple over which A is relatively type-definable).

Proof. We apply Fact 4.11 for $G := \mathrm{Aut}(\mathfrak{C})$ and A, μ from the statement of Lemma 4.12. As a result, we obtain a set $B = A \cap \sigma_1[A] \cap \dots \cap \sigma_n[A]$ for some σ_i's in $\mathrm{Aut}(\mathfrak{C})$ such that for some $l \in \mathbb{N}_{>0}$, $S := \mathrm{St}_{\mathcal{L}_{l-1}}(B)$ is generic, symmetric, and satisfies $S^{16} \subseteq AAA^{-1}A^{-1}$. Since A is relatively type-definable over some short tuple $\bar{\alpha}$, so is B, but over $\bar{a} := \bar{\alpha}\sigma_1(\bar{\alpha}) \dots \sigma_n(\bar{\alpha})$. Hence, $\mathrm{Aut}(\mathfrak{C}/\bar{a}) \cdot B = B$. Therefore, by Remark 4.10, we get that

$$\mathrm{Aut}(\mathfrak{C}/\bar{a}) \cdot S \cdot \mathrm{Aut}(\mathfrak{C}/\bar{a}) = S,$$

which means that $S = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$. To see the last thing, consider any $\sigma \in S$. We need to show that every $\tau \in \mathrm{Aut}(\mathfrak{C})$ satisfying $\mathrm{tp}(\tau(\bar{a})/\bar{a}) = \mathrm{tp}(\sigma(\bar{a})/\bar{a})$ belongs to S. For this note that there is $\tau' \in \mathrm{Aut}(\mathfrak{C}/\bar{a})$ with $\tau(\bar{a}) = \tau'\sigma(\bar{a})$. Then $\sigma^{-1}\tau'^{-1}\tau \in \mathrm{Aut}(\mathfrak{C}/\bar{a})$ and clearly $\tau = \tau'\sigma(\sigma^{-1}\tau'^{-1}\tau)$. Hence, $\tau \in S$. □(claim)

Take any $p \in \mathcal{P}$. We can write $p = p(\bar{x}, \bar{a})$ for the obvious complete type $p(\bar{x}, \bar{y})$ over $\emptyset$. Then $(A_{p,\bar{a}} \cdot A_{p,\bar{a}}^{-1})^8 \subseteq (SS^{-1})^8 = S^{16} \subseteq AAA^{-1}A^{-1} \subseteq A'$. Hence, by Corollary 4.8(ii), there is $\psi_p(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$ for which $(A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1})^8 \subseteq A'$.

Now, the complement of $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ equals $\bigcap_{p\in\mathcal{P}} A_{\neg\psi_p,\bar{a}}$ which is clearly relatively type-definable. Thus, $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}} \in \mathcal{A}$. On the other hand, $S \subseteq \bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ and S being generic implies that $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ is generic. Therefore, $\mu(\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}) > 0$.

Recall that $\tilde{\mu}$ is the $\mathrm{Aut}(\mathfrak{C})$-invariant, regular, Borel probability measure on $S_{\bar{c}}(\mathfrak{C})$ from which μ is induced. Then $\tilde{\mu}(\bigcup_{p\in\mathcal{P}}[\psi_p]) > 0$, so, by regularity, there is a compact $K \subseteq \bigcup_{p\in\mathcal{P}}[\psi_p]$ of positive measure. But K is covered by finitely many clopen sets $[\psi_p]$ one of which must be of positive measure, i.e. $\tilde{\mu}([\psi_p]) > 0$ for some $p \in \mathcal{P}$. Then $\mu(A_{\psi_p,\bar{a}}) > 0$. This implies that $Y := A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1}$ is generic (as otherwise there would exist an infinite family of pairwise disjoint left translates of $A_{\psi_p,\bar{a}}$ which would contradict the fact that μ is a left invariant probability measure), and it is clearly symmetric. By Lemma 4.7, it is also relatively type-definable. Moreover, by the choice of ψ_p, $Y^8 \subseteq A'$, so we are done. □

Corollary 4.13. *Assume* $\mathrm{Aut}(\mathfrak{C})$ *is relatively amenable. By Corollary 2.15, take the induced* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure* μ *on the Boolean algebra* $\mathcal{A}$ *generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$*. Suppose* $A \subseteq \mathrm{Aut}(\mathfrak{C})$ *is relatively type-definable and* $\mu(A) > 0$*. Then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq AAA^{-1}A^{-1}$.

Proof. Take any A' relatively definable, symmetric, and such that $AAA^{-1}A^{-1} \subseteq A'$. Put $C_0 := A'$.

By Lemma 4.12, we obtain a relatively type-definable, generic, symmetric Y such that $(Y^4)^2 \subseteq A'$. So, by Corollary 4.8, there is a relatively definable, symmetric Y' satisfying $Y^4 \subseteq Y'$ and $Y'^2 \subseteq A'$. Put $C_1 := Y'$.

Next, we apply Lemma 4.12 to Y in place of A and Y' in place of A', and we obtain a relatively type-definable, generic, symmetric Z such that $(Z^4)^2 \subseteq Y'$. So, by Corollary 4.8, there is a relatively definable, symmetric Z' satisfying $Z^4 \subseteq Z'$ and $Z'^2 \subseteq Y'$. Put $C_2 := Z'$.

Continuing in this way, we obtain a family $\{C_i : i \in \omega\}$ of relatively definable, generic, symmetric subsets of $\mathrm{Aut}(\mathfrak{C})$ such that $C_{i+1}^2 \subseteq C_i$ for every $i \in \omega$. By Corollary 4.6, $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{i\in\omega} C_i \subseteq A'$. Since A' was an arbitrary relatively definable, symmetric set containing $AAA^{-1}A^{-1}$, we get $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq AAA^{-1}A^{-1}$. □

We have now all the ingredients to prove Theorem 4.1. Our goal will be to show that for any short tuple $\bar{\alpha}$, $[\bar{\alpha}]_{E_{KP}} \subseteq [\bar{\alpha}]_{E_L}$ (which just means that the $\mathrm{Autf}_{KP}(\mathfrak{C})$-orbit of $\bar{\alpha}$ is contained in $[\bar{\alpha}]_{E_L}$).

Proof of Theorem 4.1. Recall that T being amenable means that there exists an $\mathrm{Aut}(\mathfrak{C})$-invariant, regular, Borel probability measure $\tilde{\mu}$ on $S_{\bar{c}}(\mathfrak{C})$. By Corollary 2.15, $\tilde{\mu}$ induces an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure μ on the Boolean algebra $\mathcal{A}$ generated by relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$.

Consider any $p(\bar{x}) = \mathrm{tp}(\bar{\alpha}/\emptyset) \in S(\emptyset)$ with a short subtuple $\bar{\alpha}$ of $\bar{c}$. Choose a μ-wide type $q \in S_p(\mathfrak{C})$, i.e. $\tilde{\mu}([\varphi(\bar{x}', \bar{b})]) > 0$ (equivalently, $\mu(A_{\varphi,\bar{\alpha},\bar{b}}) > 0$) for any $\varphi(\bar{x}, \bar{b}) \in q$ (where $\bar{x}' \supset \bar{x}$ is the tuple of variables corresponding to $\bar{c}$). Take a small model $M \prec \mathfrak{C}$. Replacing $\bar{\alpha}$ by a realization $\bar{\alpha}'$ of $q|_M$, we can assume that $\bar{\alpha} \models q|_M$ (because $\bar{\alpha} \equiv \bar{\alpha}'$ implies that $[\bar{\alpha}]_{E_{KP}} = [\bar{\alpha}]_{E_L}$ is equivalent to $[\bar{\alpha}']_{E_{KP}} = [\bar{\alpha}']_{E_L}$).

Consider any $\varphi(\bar{x}, \bar{\alpha}) \in q|_{\bar{\alpha}}$. Then $\mu(A_{\varphi,\bar{\alpha}}) > 0$, so, by Corollary 4.13, we conclude that $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}^{-1}A_{\varphi,\bar{\alpha}}^{-1}$. Therefore, by Corollary 4.8(i), we get

$$\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{\varphi(\bar{x},\bar{\alpha})\in q|_{\bar{\alpha}}} A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}^{-1}A_{\varphi,\bar{\alpha}}^{-1} = A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}.$$

On the other hand, Lemma 4.9 tells us that

$$A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}A_{q|_{\bar{\alpha}},\bar{\alpha}}^{-1}\bar{\alpha} \subseteq \{\bar{\beta} : d_L(\bar{\alpha}, \bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}.$$

Therefore, $[\bar{\alpha}]_{E_{KP}} = [\bar{\alpha}]_{E_L}$ has Lascar diameter at most 4. □

Theorem 4.1 is a global result. It is natural to ask whether we can extend it to a local version (as in Proposition 4.2).

Question 4.14. *Is it true that if $p(\bar{x}) \in S(\emptyset)$ is amenable, then the Lascar strong types on $p(\bar{x})$ coincide with Kim-Pillay strong types? Does amenability of $p(\bar{x})$ imply that the Lascar diameter of $p(\bar{x})$ is at most 4?*

One could think that the above arguments should yield the positive answer to these questions. The problem is that, assuming only amenability of $p(\bar{x})$, we have the induced measure μ but defined only on the Boolean algebra of relatively $\bar{\alpha}$-type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$, for a fixed $\bar{\alpha} \models p$. So, for the recursive proof of Corollary 4.13 to go through, starting from a set $A \subseteq \mathrm{Aut}(\mathfrak{C})$ relatively $\bar{\alpha}$-type-definable [where for the purpose of answering Question 4.14 via an argument as in the proof of Theorem 4.1, we can additionally assume that A is defined over $\bar{\alpha}$] of positive measure, we need to produce the desired Y also relatively $\bar{\alpha}$-type-definable [over $\bar{\alpha}$] (in order be able to continue our recursion). But this requires a strengthening of Lemma 4.12 to the version where for A relatively $\bar{\alpha}$-type-definable of positive measure one wants to obtain the desired Y which is also relatively $\bar{\alpha}$-type-definable; the variant with A and Y defined over $\bar{\alpha}$ would also be sufficient. Trying to follow the lines of the proof of Lemma 4.12, even if A is defined over $\bar{\alpha}$, Claim 1 requires a longer tuple $\bar{a}$ which produces the desired set Y which is relatively $\bar{a}$-type-definable, and this is the only obstacle to answer positively Question 4.14 via the above arguments.

Another question is whether the bound 4 on the Lascar diameters of Lascar strong types in Theorem 4.1 could be decreased. Proposition 3.11 tells us that it can be decreased to 2 under the stronger assumption of definable amenability of T.

References

[1] O. Angel, A. S. Kechris, R. Lyons, *Random orderings and unique ergodicity of automorphism groups*, J. Eur. Math. Soc. 16 (2014), 2059-2095.

[2] J. Auslander, *Minimal flows and their extensions*, North Holland, 1988.

[3] I. Ben Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. 362 (2010), 5213-5269.

[4] E. Casanovas, D. Lascar, A. Pillay, M. Ziegler, *Galois groups of first order theories*, J. Math. Log. 1 (2001), 305-319.

[5] G. Cherlin, E. Hrushovski, *Finite Structures with Few Types*, Annals of Mathematics Studies 152, Princeton University Press, Princeton and Oxford, 2003.

[6] R. M. Dudley, *Real Analysis and Probability*, Cambridge University Press, 2004.

[7] R. Elwes and D. Macpherson, *A survey of asymptotic classes and measurable structures.* In: Model Theory with Applications to Algebra and Analysis, Vol. 2, Ed. by: Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie, Cambridge University Press, 2008, 125-160.

[8] D. H. Fremlin, *Measure Theory*, Vol. 4, Torres Fremlin, 2003.

[9] J. Gismatullin, D. Penazzi, A. Pillay, *On compactifications and the topological dynamics of definable groups*, Ann. Pure Appl. Logic 165 (2014), 552-562.

[10] E. Glasner, *Proximal flows*, Lecture Notes in Mathematics 517, Springer Verlag, 1976.

[11] D. Haskell, E. Hrushovski, D. Macpherson, *Stable domination and independence in algebraically closed valued fields.* Lecture Notes in Logic, 30. Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2008. xii+182 pp. ISBN: 978-0-521-88981-0

[12] E. Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. 25 (2012), 189-243.

[13] E. Hrushovski, K. Krupiński, A. Pillay, *Amenability, connected components, and definable actions*, preprint.

[14] E. Hrushovski, Y. Peterzil, A. Pillay, *Groups, measures and the NIP*, J. Amer. Math. Soc. 21 (2008), 563-596.

[15] E. Hrushovski, A. Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc. 13 (2011), 1005-1061.

[16] A. A. Ivanov, *Strongly determined types and G-compactness*, Fund. Math. 191 (2006), 227-247.
[17] A. A. Ivanov and D. Macpherson, *Strongly determined types*, Ann. Pure Appl. Logic 99 (1999), 197-230.
[18] A. S. Kechris, V.G. Pestov, S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. 15 (2005), 106-189.
[19] H. J. Keisler, *Measures and forking*, Ann. Pure Appl. Logic 34 (1987), 119-169.
[20] B. Kim, A. Pillay, *Simple theories*, Ann. Pure Appl. Logic 88 (1997), 149-164.
[21] K. Krupiński, J. Lee, S. Moconja, *Ramsey theory and topological dynamics for first order theories*, Trans. Amer. Math. Soc. 375 (2022), 2553-2596.
[22] K. Krupiński, L. Newelski, *On bounded type-definable equivalence relations*, Notre Dame J. Form. Log. 43 (2002), 231-242.
[23] K. Krupiński, L. Newelski, P. Simon, *Boundedness and absoluteness of some dynamical invariants*, J. Math. Log. 19 (2019), 1950012 (55 pages).
[24] K. Krupiński, A. Pillay, *Amenability, definable groups, and automorphism groups*, Adv. Math. 345 (2019), 1253-1299.
[25] K. Krupiński, A. Pillay, T. Rzepecki, *Topological dynamics and the complexity of strong types*, Isr. J. Math. 228 (2018), 863-932.
[26] D. Lascar, *The category of models of a complete theory*, J. Symbolic Logic 47 (1982), 249-266.
[27] D. Lascar, A. Pillay, *Hyperimaginaries and automorphism groups*, J. Symbolic Logic 66 (2001), 127-143.
[28] J.-C. Massicot, F. O. Wagner *Approximate subgroups*, Journal de l'École Polytechnique 2 (2015), 55-63.
[29] L. Newelski, *The diameter of a Lascar strong type*, Fund. Math. 176 (2003), 157-170.
[30] P. Simon, *A guide to NIP theories (Lecture Notes in Logic)*, Cambridge University Press, 2015.
[31] F. Wagner, *Simple theories*, Kluwer Academic Publishers, 2000.
[32] M. Ziegler, *Introduction to the Lascar Group*, In: Tits Buildings and the Model Theory of Groups, Ed. by Katrin Tent, London Mathematical Society Lecture Note Series, Cambridge University Press, 2002, 279-298.

Email address, E. Hrushovski: Ehud.Hrushovski@maths.ox.ac.uk

(E. Hrushovski) Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter (550), Woodstock Road, Oxford OX2 6GG, UK

Email address, K. Krupiński: kkrup@math.uni.wroc.pl

(K. Krupiński) Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki, 50-384 Wrocław, Poland

ORCID (K. Krupiński): 0000-0002-2243-4411

Email address, A. Pillay: apillay@nd.edu

(A. Pillay) Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame, IN 46556, USA