On ordered minimal structures

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Abstract

We investigate minimal first-order structures and consider interpretability and definability of orderings on them. We also prove the minimality of their canonical substructures.

The motivation for this work comes from several papers by Tanović concerning the Pillay's Conjecture [1], which states that every countable structure in a countable language has infinitely many nonisomorphic, countable extensions. Tanović proved the conjecture in [2], and the machinery of the proof relied on the notion of C-sequences, introduced in [5]. Results on the C-sequences emerged from analysis of particular orders defined in the original structure. In [3, 4], the earlier papers, a related and more general analysis has been done concerning minimal ordered structures. That analysis produced further results, such as a partial answer to the Kueker's Conjecture [6] and reduction of the Podewski's Conjecture to a simpler case [7]. The results from that analysis, recalled and used throughout this paper, provided a partial characterization for such structures, and left the following:

Question 1. *Does every minimal, partially ordered structure with arbitrarily long chains interpret a minimal infinite linear order?*

The question is asked in a general context and its essentially identical variants have been repeated in [5] and [2]. It is the aim of this paper to elaborate on the subject.

The main results of the paper are Theorem 7 and Theorem 8, explaining the structure of a minimal structure $(M, <)$.

We begin by setting up the notation and recalling the fundamental results from [3, 4]. All structures in the paper are models of an arbitrary first-order theory in a language extending the language $\{<\}$ where $<$ is interpreted as an ordering.

Definition 2. *Let $(M, <)$ be a minimal ordered structure. Let p be the unique non-algebraic type over M. We define*

$$\begin{aligned} L_<(M) &= \{m \in M : (m < x) \in p\}, \\ U_<(M) &= \{m \in M : (m > x) \in p\}, \\ I_<(M) &= \{m \in M : (m \perp x) \in p\}. \end{aligned}$$

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The order relation in the subscript will be omitted if the context is clear.

The following result appears in [3, Proposition 1.1, Proposition 1.2] and [4, Theorem 2].

Theorem 3. *Let $(M,<)$ be a minimal ordered structure with an infinite chain. Then*

(1) M is countable.

(2) $I_<(M)$ is finite.

(3) Assuming $(M,<)$ has an increasing infinite chain, it falls into one of the following two types:

$$\begin{array}{lll} \mathrm{Type}(\omega) & \textit{where} & M = L_<(M) \cup \textit{ a finite set;} \\ \mathrm{Type}(\omega+\omega^*) & \textit{where} & M = L_<(M) \cup U_<(M) \cup \textit{ a finite set} \end{array}$$

(here both $L_<(M)$ and $U_<(M)$ are infinite).

The theorem gives further description of the structure: if $(M,<)$ is of Type(ω) then $(L_<(M),<)$ has no maximal elements, is directed upwards and does not contain chains of order type $\omega+1$. If $(M,<)$ is of Type$(\omega+\omega^*)$ then $(L_<(M),<)$ has no maximal elements, is directed upwards and does not contain chains of order type $\omega+1$ while $(U_<(M),<)$ has no minimal elements, is directed downwards and does not contain chains of order type $1+\omega^*$, where ω^* is the ω with reverse order relation.

The last result shows that the type of a structure is well defined in a class of interdefinable minimal orders with infinite, increasing chains:

Proposition 4 ([4], Theorem 3)**.** *Let $\mathbb{M} = (M,\ldots)$ be a minimal structure. The following are equivalent:*

(1) There exists a definable $<$ such that $(M,<)$ is of Type(ω).

(2) $\mathbb{M}$ is ordered of Type(ω) *with respect to any ordering with an infinite increasing chain.*

The following is easily proven (similarly to the König's Lemma):

Fact 5. *Let $(M,<)$ be minimal. Then $<$ has an infinite chain iff $<$ has chains of arbitrary length.*

From now on, we assume that all considered structures $(M,<)$ have an infinite chain.

By Theorem 3(2), given a minimal $(M,<)$ it is always possible to define an order $<'$ such that $I_{<'}(M)=\emptyset$. Thus for the remainder of this paper we additionaly assume $I_<(M)=\emptyset$.

Now assume that $(M,<)$ is minimal of Type(ω). Then it is easy to see that $<$ is the transitive closure of the successor relation S induced by it: for every $a<b$ there are only finitely many elements c such that $a<c<b$. Given that

every definable antichain must be finite, we can further modify $<$ so it has the smallest element. Now $(M,<)$ can be described as a rooted, directed acyclic graph: we put an egde between a and b whenever aSb.

Note that for any $n \in \omega$ the set L_n of elements that are (in the sense of the graph) n edges away from the root is $\emptyset$-definable. This gives the following:

Fact 6. $M \subset \mathrm{acl}(\emptyset)$.

It is easy to see that a similar description (as well as the fact) holds for $\mathrm{Type}(\omega+\omega^*)$ structures as well, in which case $<$ is the transitive closure of the induced successor relation together with "$x<y$" for all $x \in L(M), y \in U(M)$.

So at glance, a minimal $(M,<)$ of $\mathrm{Type}(\omega+\omega^*)$ consists of two "halves" that have no other obvious relations between them (other than $L(M) < U(M)$). Given such a "half", say $L(M)$, it is natural to ask about the fashion in which it (together with the order relation restricted to it) is embedded in the whole structure. The main results of the paper are the following Theorem 7 and Theorem 8.

Theorem 7. *Let* $\mathbb{M} = (M,<)$ *be of* $\mathrm{Type}(\omega+\omega^*)$ *and* $\mathbb{L} = (L(M), <\restriction_{L(M)})$. *Then*

(1) $\mathbb{L}$ *is minimal,*

(2) For any $A \subset M^n$ *definable in* $\mathbb{M}$*, the set* $A \cap L(M)^n$ *is definable in* $\mathbb{L}$*,*

(3) For any $B \subset L(M)^n$ *definable in* $\mathbb{L}$*, there is an* $A \subset M^n$ *definable in* $\mathbb{M}$ *such that* $A \cap L(M)^n = B$.

In particular the theorem asserts that $\mathbb{L}$ is "stably embedded" in $\mathbb{M}$.

In the context of interpretable orderings, having established Theorem 7(1), we will also prove:

Theorem 8. *Let* $\mathbb{M} = (M,<)$ *be of* $\mathrm{Type}(\omega+\omega^*)$ *and* $\mathbb{L} = (L(M), <\restriction_{L(M)})$. *If* $\mathbb{M}$ *interprets an infinite order, then* $\mathbb{L}$ *interprets an infinite suborder of it.*

Note that we do not assume the minimality or linearity of the interpreted order.

In order to prove the theorems, we begin with a crucial lemma. Before we state it, we need to set up some additional notation.

Definition 9. *(i) Given an* $n \in \omega$ *and any* $A \subset M$*, an* (n,A)*-type is a complete type over* A *restricted to the formulas of quantifier rank* $\leq n$.

(ii) With n, A *as above and* $\bar{a}, \bar{b} \in M^m$*, we write* $\bar{a} \equiv_{n,A} \bar{b}$ *whenever* $\bar{a}$ *and* $\bar{b}$ *satisfy the same* (n,A)*-type.*

Note that for any n and any finite A, the set of all (n,A)-types is finite.

Lemma 10. *Let* $(M,<)$ *be minimal of* $\mathrm{Type}(\omega+\omega^*)$. *We write* L *for* $L(M)$ *and* U *for* $U(M)$. *For any* $n \in \omega$,

F_n) For each $k, l \in \omega$, $\bar{a} \in L^k$ and $\phi(x, \bar{y}, \bar{z}) \in \mathcal{L}(U)$ with $\mathrm{qr}(\phi) \le n, |\bar{y}| = k, |\bar{z}| = l$, *there is a finite $S \subset L$ such that for each $\bar{b} \in U^l$,*

$$\phi^M(M, \bar{a}, \bar{b}) \subset L \Rightarrow \phi^M(M, \bar{a}, \bar{b}) \subset S.$$

S_n) For each $k, l \in \omega$ there is a finite $S_{n,k,l} \subset L$ such that for each $\bar{b}_1, \bar{b}_2 \in U^l$ and $\bar{a} \in L^k$,

$$\bar{b}_1 \equiv_{n,S_{n,k,l}} \bar{b}_2 \Rightarrow \bar{b}_1\bar{a} \equiv_n \bar{b}_2\bar{a}.$$

Proof. First, we note that it is enough to prove the Lemma with $\phi(x, \bar{y}, \bar{z}) \in \mathcal{L}$ in the statement of F_n as the full statement follows from it. So we replace $\mathcal{L}(U)$ there with $\mathcal{L}$ for the remainder of the proof. We proceed by simultaneous induction, proving the following:

(1) S_0 holds,

(2) S_n implies F_n,

(3) $S_n + F_n$ imply S_{n+1}.

(1) Let $S_{0,k,l} = \emptyset$. Assume $\bar{b}_1 \equiv_{0,\emptyset} \bar{b}_2$. For any $b_1 \in \bar{b}_1, b_2 \in \bar{b}_2, a \in \bar{a}$ we have $a < b_1, a < b_2$. So $\bar{b}_1\bar{a} \equiv_{0,\emptyset} \bar{b}_2\bar{a}$.
(2) Assume S_n and fix $\phi(x, \bar{y}, \bar{z})$ and $\bar{a} \in L^k$. By S_n there is a finite $S_{n,k+1,l}$ such that for each $a \in L$ and $\bar{b}_1, \bar{b}_2 \in U^l$ with $\bar{b}_1 \equiv_{n,S_{n,k+1,l}} \bar{b}_2$ we have

$$\models \phi(a, \bar{a}, \bar{b}_1) \iff \phi(a, \bar{a}, \bar{b}_2).$$

So whenever the set $\phi(M, \bar{a}, \bar{b})$ is contained in L (and therefore is finite), that set depends only on the $(n, S_{n,k+1,l})$-type of $\bar{b}$. Since there are only finitely many of such types, we conclude that

$$\bigcup\{\phi(M, \bar{a}, \bar{b}) : \bar{b} \in U^l, \phi(M, \bar{a}, \bar{b}) \subset L\}$$

is a finite union of finite sets. We define S to be this union.
(3) We construct $S_{n+1,k,l}$, stipulating that it contains the already defined sets $S_{n,k+1,l} \cup S_{n,k,l+1}$ and enlarging it as needed. We aim to show

$$\bar{b}_1 \equiv_{n+1,S_{n+1,k,l}} \bar{b}_2 \Rightarrow \bar{b}_1\bar{a} \equiv_{n+1} \bar{b}_2\bar{a}.$$

We prove that for each $c \in M$ there is a $c' \in M$ such that $\bar{b}_1\bar{a}c \equiv_n \bar{b}_2\bar{a}c'$. We distinguish three cases.

Case 1. $c \in L$. Assume $\bar{b}_1 \equiv_{n+1,S_{n+1,k,l}} \bar{b}_2$. In particular, $\bar{b}_1 \equiv_{n,S_{n,k+1,l}} \bar{b}_2$ and by the induction hypothesis, $\bar{b}_1c\bar{a} \equiv_n \bar{b}_2c\bar{a}$.

Case 2. $c \in U$ and there is a $d \in U$ such that $\bar{b}_1c \equiv_{n,S_{n+1,k,l}} \bar{b}_2d$. Then, as $S_{n+1,k,l} \supset S_{n,k,l+1}$, by the induction hypothesis again we get $\bar{a}\bar{b}_1c \equiv_n \bar{a}\bar{b}_2d$.

Case 3. $c \in U$ and for each $d \in M$,

$$\bar{b}_1 c \equiv_{n,S_{n+1,k,l}} \bar{b}_2 d \Rightarrow d \in L.$$

Here, we prove that for a sufficiently large (yet finite) $S_{n+1,k,l}$, this case leads to a contradiction. So, observe that the assumption implies that $d\bar{b}_2$ satisfies some $(m, S_{n,k,l+1})$-type p.
Take any $\phi(x,\bar{y}) \in p$ such that $\phi(M,\bar{b}_2) \subset L$. By F_n there is a finite $S = S_\phi$ such that whenever $\phi(M,\bar{b}) \subset L$ for some $\bar{b} \in U^l$, we also have $\phi(M,\bar{b}) \subset S$. There are only finitely many formulas in p, so let $S_{n+1,k,l} \supset \bigcup_\phi S_\phi$, where the union ranges over all formulas $\phi \in p$ with realizations contained in L. The union depends only on $S_{n,k,l+1}$, so $S_{n+1,k,l}$ is well-defined. As d certainly belongs to this union, $d \in S_{n+1,k,l}$ and $\bar{b}_1 c \not\equiv_{n,S_{n+1,k,l}} \bar{b}_2 d$, a contradiction. □

Proof of Theorem 7. (1) Follows from the statement of Lemma 10.
(2) Write L for $L(M)$ and U for $U(M)$. Let $\Sigma \subset \mathcal{L}(M)$ be the set of formulas $\phi(\bar{x})$ such that there is $\phi'(\bar{y}) \in \mathcal{L}(L)$ with $\phi^{\mathbb{M}}(L^n) = \phi'^{\mathbb{L}}(L^n)$. It is easy to see that Σ contains all atomic formulas (recall that $a < b$ whenever $a \in L, b \in M$). It is clearly closed under taking disjunctions and negations. We prove that $\Sigma = \mathcal{L}(M)$ by induction on quantifier rank of formulas. So let

$$\phi(\bar{x}) = \exists y \psi(\bar{x}, y),$$

with $|\bar{x}| = n$ and $\psi(\bar{x}, y)$ belonging to Σ along with all formulas of the same quantifier rank. In particular, all instances of ψ belong to Σ.

Consider all tuples $\bar{l} \in L^n$ such that $\psi(\bar{l}, M) \subset U$. By Lemma 10 (with U and L swapped), there is a finite U_0 such that $\psi(\bar{l}, M) \subset U \Rightarrow \psi(\bar{l}, M) \subset U_0$. Let

$$\chi(\bar{x}, y) = \bigvee_{u \in U_0} \psi(\bar{x}, u) \vee \psi(\bar{x}, y).$$

One checks that for any $\bar{l} \in L^n$,

$$\mathbb{M} \models \phi(\bar{l}) \iff \exists a \in L \; \mathbb{M} \models \chi(\bar{l}, a).$$

As $\chi(\bar{x}, y)$ is a disjunction of formulas from Σ, there is $\chi'(\bar{x}, y) \in \mathcal{L}(L)$ such that $\chi^{\mathbb{M}}(L^n, L) = \chi'^{\mathbb{L}}(L^n, L)$. Now clearly $\exists y \chi'(\bar{x}, y)$ defines in $\mathbb{L}$ the set $\phi^{\mathbb{M}}(L^n)$.
(3) Again, write L for $L(M)$ and U for $U(M)$. We proceed in a similar manner as before. So let $\Sigma \subset \mathcal{L}(L)$ be the set of formulas such that their realizations in $\mathbb{L}$ are traces of sets definable in $\mathbb{M}$. Again it is easy to see that Σ contains atomic formulas and is closed under boolean combinations. Let

$$\phi(\bar{x}) = \exists y \psi(\bar{x}, y),$$

with $|\bar{x}| = n$ and $\psi(\bar{x}, y)$ belonging to Σ. So there is $\psi'(\bar{x}, y) \in \mathcal{L}(M)$ such that $\psi'^{\mathbb{M}}(L^n, L) = \psi^{\mathbb{L}}(L^n, L)$. By Lemma 10 there is a finite U_0 such that for each $\bar{l} \in L^n$ we have $\psi'^{\mathbb{M}}(\bar{l}, M) \subset U \Rightarrow \psi'^{\mathbb{M}}(\bar{l}, M) \subset U_0$. Let

$$\chi'(\bar{x}, y) = \psi'(\bar{x}, y) \wedge \bigwedge_{u \in U_0} y \neq u.$$

Then $\exists y \chi'(\bar{x}, y)$ witnesses $\phi(\bar{x}) \in \Sigma$. □

Turning attention to Theorem 8, consider any order interpretable in $\mathbb{M}$, that is a definable equivalence relation on M^n and a definable ordering on its classes. By Theorem 7, their restrictions to $L(M)^n$ are definable in $\mathbb{L}$, giving an interpretable order in $\mathbb{L}$. The following lemma guarantees that this order has infinitely many elements, proving Theorem 8.

Lemma 11. *Let $(M,<)$ be minimal of* Type$(\omega+\omega^*)$ *and E be a definable equivalence relation on M^n with infinitely many classes. Write U for $U(M)$ and L for $L(M)$. Then at least one of the following holds:*

(1) $E\restriction_{U^{2n}}$ and $E\restriction_{L^{2n}}$ have infinitely many classes.

(2) There is an $a \in M$ and $i < n$ such that E restricted to

$$(\underbrace{M \times \ldots \times M \times \{a\} \times M \times \ldots \times M}_{n \text{ times}})^2,$$

where $\{a\}$ is on the i-th axis, has infinitely many classes.

Proof. Assume that 2) does not hold and (aiming for a contradiction) that $E\restriction_{U^{2n}}$ has finitely many classes. We will show that E also has finitely many classes. M^n is the union of sets of the form $H_0 \times \ldots \times H_{n-1}$ where each of H_i's is either U or L. For a set X of this form, let $|X|$ be number of L's that appear in the product. We show the following by induction on k:

$(*_k)$ Let X be of the form as above and $|X| = k$. Then $E\restriction_{X^2}$ has finitely many classes.

We assumed $*_0$. Now take any X with $|X| = k$ and assume $*_{k'}$ for all $k' < k$. Without loss of generality $X = U^{n-k} \times L^k$. There is a single formula $\phi(\bar{x})$ with $|\bar{x}| = n$ saying "$\bar{x}$ is not E-equivalent to any $\bar{m} \in X$ with $|X| < k$". Consider the formula

$$\psi(x, \bar{y}) = \exists x_{n-k}, x_{n-k+1}, \ldots, x_{n-1} \phi(\bar{y}, x, x_{n-k}, \ldots x_{n-1}).$$

Whenever we consider an instance $\psi(x, \bar{u})$ for some $\bar{u} \in U^{n-k-1}$, all x's satisfying this instance must belong to L, since the tuples satisfying $\phi(\bar{x})$ have at most $n-k-1$ elements of U. So we can apply Lemma 10 to $\psi(x, \bar{y})$: there is a finite $S \subset L$ such that for any $\bar{u} \in U^{n-k-1}$, $\phi(M, \bar{u}) \subset S$. Unrolling the definition of ψ we have that all elements of $U^{n-k} \times (L \setminus S) \times L^{k-1}$ satisfy ϕ, i.e. they are contained in finitely many of the E-classes. By the assumption that 2) does not hold, for each $a \in S$ the set $U^{n-k} \times \{a\} \times L^{k-1}$ is also contained in a finite union of E-classes. But

$$X = \bigcup_{a \in S} U^{n-k} \times \{a\} \times L^{k-1} \cup U^{n-k} \times (L \setminus S) \times L^{k-1},$$

and we are done. $\square$

Theorem 8 should be viewed as an attempt to deal with Question 1. In dealing with Podewski's conjecture, only structures of Type(ω) are concerned [7]. In both Podewski's and Kueker's conjectures, only definable orders are of the matter [6, 7]. It would be beneficial to have equivalences of the kind "no order definable in $\mathbb{L}$ iff no order interpretable in $\mathbb{M}$" which would restrict the area in which any counterexample has to be found, or provide a way to produce definable orders having only an interpretable one in a structure of different type. An example of such equivalence can be proven (here we do not make any assumptions on M other than stated in the Proposition):

Proposition 12. *Let M be a minimal ordered structure that interprets $(\omega, \leq)$. Then there exists a definable equivalence relation E on M and a definable order $\preceq$ on M/E such that $(M/E, \preceq) \cong (\omega, \leq)$.*

Proof. Let $n \in \mathbb{N}$ be the least such that there is a definable equivalence relation E' on M^n and a definable order $\leq'$ on M^n/E' with $(M^n/E', \leq') \cong (\omega, \leq)$. Assume that $n > 1$. We head for a contradiction. For each $m \in M$ let

$$C(m) = \{(m, m_2, m_3, \ldots, m_n)_{E'} \in M^n/E' : m_2, m_3, \ldots, m_n \in M\}.$$

Under the isomorphism $(M^n/E', \leq') \cong (\omega, \leq)$ the set $C(m)$ corresponds to a subset of ω.

Case 1 For each $m \in M$, $C(m)$ is finite. Let k_m be the greatest element of such a subset. We show that a desired definable relation and a definable ordering on classes can be found in the M itself.

Let $\preceq$ be defined on M by

$$a \preceq b \iff \exists \bar{y} \forall \bar{x} \forall \bar{y}' \; ((a, \bar{x})_{E'} \leq' (b, \bar{y})_{E'} \wedge (b, \bar{y}')_{E'} \leq' (b, \bar{y}))_{E'},$$

and let E be defined by

$$a \, E \, b \iff \exists \bar{y} \exists \bar{x} \forall \bar{y}' \; ((a, \bar{x}) \, E' \, (b, \bar{y}) \wedge (b, \bar{y}') \leq' (b, \bar{y}) \wedge (a, \bar{y}') \leq' (a, \bar{x})).$$

Immediately from the definitions, we have $a \preceq b$ iff the greatest element of $C(b)$ is larger than any element of $C(a)$ and we have aEb iff the greatest elements of $C(a)$ and $C(b)$ are equal. We have

$$\begin{aligned} a \preceq b &\iff k_a \leq k_b, \\ a \, E \, b &\iff k_a = k_b. \end{aligned}$$

It is now sufficient to prove that $\{k_m \in \omega : m \in M\}$ is an infinite subset of ω. But this follows from the fact that $\bigcup\{C(m) : m \in M\} = M^n/E$: there can be no uniform bound on the greatest element of $C(m)$.

Case 2 There is $m_0 \in M$ such that $C(m_0)$ is infinite. We construct the desired relations on M^{n-1}, contradicting the minimality of n.

Let $\preceq$ be defined on M^{n-1} by

$$\bar{a} \preceq \bar{b} \iff (m_0, \bar{a}) \leq' (m_0, \bar{b}),$$

and let E be defined by

$$\bar{a}\, E\, \bar{b} \iff (m_0, \bar{a})\, E'\, (m_0, \bar{b}).$$

It is easy to see that $(M^{n-1}/E, \preceq) \cong (\omega, \leq)$. $\square$

We conclude the paper with a remark regarding minimal structures of SOP theories. Given an SOP theory T and a formula $\phi(\bar{x})$ (without parameters) with the strict order property, it is natural to ask whether any minimal $M \models T$ can be ordered with infinite chains in a definable way, involving the formula $\phi(\bar{x})$. It is true when $|x| = 1$: $\phi(x)$ defines an order on M and since it has chains of arbitrary length in a monster, it also has chains of arbitrary length in M. Thus it has an infinite chain there.

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