Dopełnialne podprzestrzenie $C(K \times L)$

Grzegorz Plebanek

Uniwersytet Wrocławski

współautorzy
Jakub Rondoš and **Damian Sobota** (KGRC, Wien)

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For the zero-dimensional space K:

The space $C(K)$ is Grothendieck iff for every bounded sequence of (signed regular Borel measures of finite variation) measures μ_n

$$\Big(\forall A \in \mathrm{clop}(K)\Big) \lim_n \mu_n(A) = 0 \Longrightarrow \Big(\forall B \in Bor(K)\Big) \lim_n \mu_n(B) = 0.$$

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Typical examples of $C(K)$ Grothenideck spaces are $C(K)$ where K is zero-dimensional and the algebra $\mathrm{clop}(K)$ has some weak 'sequential completeness property', see Koszmider & Shelah (2013) and González & Kania (2021).

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Alspach and Galego (2011):

Does $C(\beta\omega \times \beta\omega)$ contain complemented copies of other separable (infinite-dimensional) Banach spaces?

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- $C(L)$ is isomorphic to $C[0,1]$ whenever L is uncountable compact metrizable space;
- there are uncountably many pairwise non-isomorphic $C(L)$ spaces where L is compact and countable.

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Main Theorem

Suppose that compact spaces K_1, K_2 can be continuously mapped onto some compact topological group G.
Then $C(K_1 \times K_2)$ contains a complemented **isometric** copy of the space $C(G)$.

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Proof.

$$T : C(K) \to C(L), \quad Tf(y) = \int_K f \, \mathrm{d}\mu_y,$$

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- $\varphi[\mu](A) = \mu\big(\varphi^{-1}[A]\big)$ for $A \in Bor(L)$.
- $\triangle$-density: For every $B \in Bor(K)$ and $\varepsilon > 0$ there is $S \in \Sigma$ such that $\mu(V \triangle S) < \varepsilon$.

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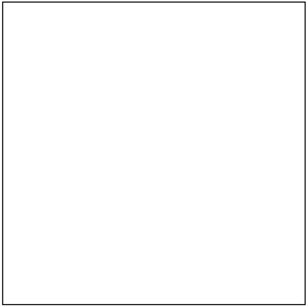

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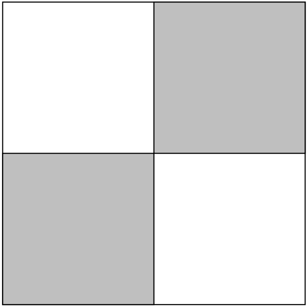

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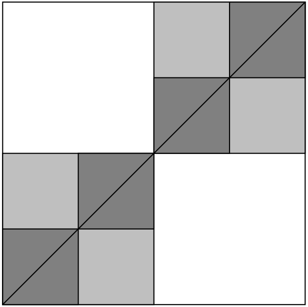

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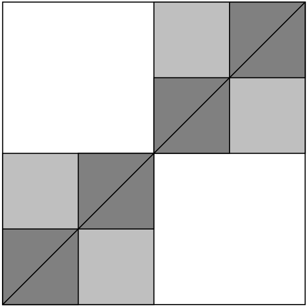

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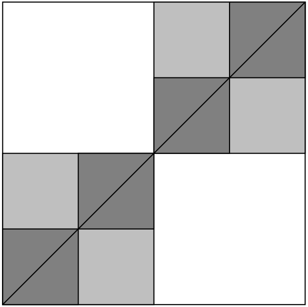

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We have $\nu_n \to \nu$, where ν denotes λ put on the diagonal.

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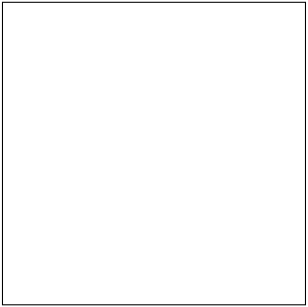

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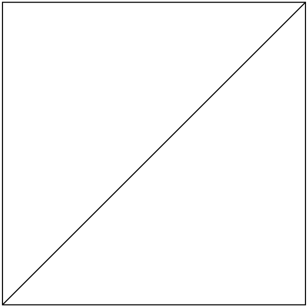

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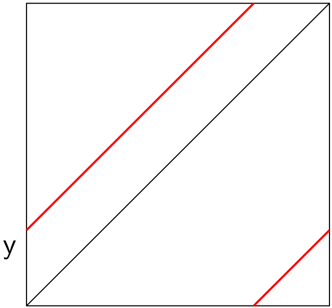

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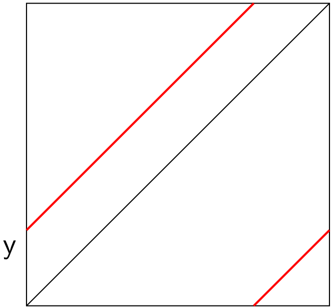

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ν^y is the limit measure on the shifted diagonal $\{(x, x \oplus y) : y \in G\}$.

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On the other hand, Marciszewski (2008) proved that
$C(K) = C[0,1] \oplus C(K)$.

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If K is zero-dimensional and the algebra $\mathrm{clop}(K)$ admits a Boolean homomorphism onto a free product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of nonatomic Boolean algebras then $C(K)$ has a complemented subspace isomorphic to $C[0,1]$.

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$K = \big((0,1] \times \{0\}\big) \cup \big([0,1) \times \{1\}\big)$.
The projection $\varphi : K \to [0,1]$ defines an isometric embedding of $C[0,1]$ onto an uncomplemented subspace X of $C(K)$. In fact there is no complemented separable superspace $Y \supseteq X$, see Kalenda and Kubiś (2012).
On the other hand, Marciszewski (2008) proved that
$C(K) = C[0,1] \oplus C(K)$.
The above corollary does not work here: K does not contain a product of two non-scattered compacta, see e.g. Martínez Cervantes and GP (2019).

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Does $C(\beta\omega \times \beta\omega)$ contains a complemented copy of $C(K)$ for every separable K?

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Then extend μ_0 to a Borel measure preserving the density condition.