

# Strictly Positive Measures on Boolean Algebras

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## Abstract

We investigate strictly positive finitely additive measures on Boolean algebras and strictly positive Radon measures on compact zerodimensional spaces. The motivation is to find a combinatorial characterisation of Boolean algebras which carry a strictly positive finitely additive finite measure with some additional properties, such as separability or nonatomicity. A possible consistent characterisation for an algebra to carry a separable strictly positive measure was suggested by Talagrand in 1980, which is that the Stone space  $K$  of the algebra satisfies that its space  $M(K)$  of measures is weakly separable, equivalently that  $C(K)$  embeds into  $l^\infty$ . We show that there is a *ZFC* example of a Boolean algebra (so of a compact space) which satisfies this condition and does not support a separable strictly positive measure. However, we use this property as a tool in a proof which shows that under  $MA + \neg CH$  every atomless ccc Boolean algebra of size  $< \mathfrak{c}$  carries a *nonatomic* strictly positive measure. Examples are given to show that this result does not hold in *ZFC*. Finally, we obtain a characterisation of Boolean algebras that carry a strictly positive nonatomic measure in terms of a chain condition, and we draw the conclusion that under  $MA + \neg CH$  every atomless ccc Boolean algebra satisfies this stronger chain condition.

## 0 Introduction.

All terms necessary to understand this paper are given in the preliminaries. Some terms are used only in the introduction and are given without definition.

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A strictly positive measure on a Boolean algebra is a finitely or countably additive (depending on the context) measure which assigns positive value to every nonzero element of the algebra. Recognising by means of a combinatorial criterion which algebras carry such a measure has been a topic of continuous interest at least since von Neumann asked in 1937 in *The Scottish Book* (see [21]) whether every ccc weakly distributive  $\sigma$ -complete Boolean algebra is a measure algebra. A criterion suggested by Maharam was the existence of a strictly positive continuous submeasure on the algebra, and the problem if these conditions are sufficient was known as the Control Measure Problem. Much progress has been achieved recently in recognising which complete ccc Boolean algebras do carry a continuous submeasure (see [3], [27], [10]), finally enabling Todorćević in [29] to formulate the criterion that a complete Boolean algebra carries a strictly positive continuous submeasure iff it is weakly distributive and satisfies the  $\sigma$ -finite chain condition. These results reduced von Neumann's problem to the Control Measure Problem, whose negative solution however was recently obtained by Talagrand ([26], taking Farah's [9] as a basis). In particular, Talagrand's result answers negatively the problem of von Neumann.

Talagrand's results bring us back to square one as far as recognising combinatorially which  $\sigma$ -complete ccc algebras carry a countably additive strictly positive measure. Our emphasis however will be on *finitely* additive strictly positive measures. Restricting our attention to this type of measures we no longer need to concentrate only on  $\sigma$ -complete Boolean algebras. On the other hand, finitely additive strictly positive measures are already a wide enough class of measures. For example, a weakly distributive Boolean algebra carries a countably additive strictly positive measure iff it carries a finitely additive strictly positive measure (see [13]) and every Radon (so countably additive) measure supported by a zerodimensional compact space is the natural extension to its Stone space of a strictly positive finitely additive measure on a Boolean algebra. An important point is that there is already a combinatorial criterion on the Boolean algebras that carry a strictly positive finitely additive measure, namely Kelley's criterion from [19], see the preliminaries. The purpose of our work is to investigate possible improvements of this criterion which will enable us to recognise when the Boolean algebra supports a strictly positive finitely additive measure with additional features, such as being separable or nonatomic. Separable measures are those for which there is a countable subset of the algebra which approximates all elements of the algebra arbitrarily close in the measure. Such measures are considered as the most natural ones (see e.g. [25]), in particular because the most common examples of measures (such as the Lebesgue measure on the unit interval) have this property. Separable compact spaces support a somewhat trivial separable Radon measure, namely a weighted sum of the point-weight measures for the points of a countable dense set. It is more difficult to support a measure in which all points have measure zero. Nonatomic measures are the analogue of this notion for Boolean algebras. Of course, the existence of a countably additive strictly positive measure on a Boolean algebra can be viewed as a property that strengthens the property of supporting just a finitely additive strictly positive measure, so this line of research may be viewed also in

the light of trying to add something to the conditions by von Neumann and Maharam to find the conditions that actually do characterise measure algebras. Here we however concentrate on the two properties we discussed above, separability and nonatomicity.

A condition for the existence of a strictly positive separable measure on a Boolean algebra was suggested by Talagrand in [25]. We call this condition approximability. Talagrand showed that under  $CH$  approximability is not sufficient for a Boolean algebra to carry a separable strictly positive measure. We show in §1 that there is a  $ZFC$  counterexample. The algebra showing this has size  $\mathfrak{c}$ . On the other hand, we construct for every  $\sigma$ -finite cc Boolean algebra a specific ccc forcing which makes it have a strictly positive measure and have size  $\leq \mathfrak{c}$  (§2). A forcing notion with these properties is already known (see [14]), but our forcing has an additional feature related to approximability, which allows us to show that under  $MA+\neg CH$  every atomless ccc Boolean algebra of size  $< \mathfrak{c}$  carries a nonatomic separable strictly positive measure. (The  $\sigma$ -finite cc condition mentioned above is the notion introduced by Horn and Tarski in [17] and it is the same condition which appears in the above mentioned characterisation by Todorčević.) On the other hand, we show that it is consistent to have a ccc Boolean algebra of size  $< \mathfrak{c}$  with no strictly positive measure, or that there is a ccc Boolean algebra of size  $< \mathfrak{c}$  with a strictly positive measure but not a separable such measure.

We were not able to characterise combinatorially which Boolean algebras carry a strictly positive separable measure. However we did obtain a combinatorial characterisation of those Boolean algebras that carry a nonatomic strictly positive measure. The characterisation takes form of a chain condition (Theorem 2.9). A corollary of this and the theorems mentioned above is the equivalence under  $MA + \neg CH$  between the ccc property of an atomless Boolean algebra of size  $< \mathfrak{c}$  and the stronger chain condition given in Theorem 2.9.

An excellent article surveying many ccc conditions in topology is Todorčević's [28].

## 0.1 Preliminaries.

All spaces considered here are Hausdorff. Boolean algebras are assumed to be fields of sets and so  $\wedge$  and  $\cap$  are used interchangeably, as well as  $\vee$  and  $\cup$  and  $\leq$  and  $\subseteq$ . We shall use the usual convention that  $b^0 = b$  and  $b^1 = b^c$  for any element  $b$  of a Boolean algebra.

**Definition 0.1** *Let  $\mathfrak{B}$  be a Boolean algebra.*

(1)  $\mathfrak{B}$  has the intersection number  $\geq \alpha$  if for every  $n < \omega$  and every  $n$  positive elements of  $\mathfrak{B}$ , possibly with repetitions, there are at least  $\alpha \cdot n$  among them which have a nonempty intersection. The intersection number of a Boolean algebra  $\mathfrak{B}$  is the sup of all  $\alpha$  such that the intersection number of  $\mathfrak{B}$  is  $\geq \alpha$ . We denote this by  $\text{int}(\mathfrak{B})$ .

$\mathfrak{B}$  satisfies the Kelley condition if it is a countable union of subsets with positive intersection number.

(2)  $\mathfrak{B}$  has the  $\sigma$ -finite cc if it is a countable union of subsets none of which has an infinite antichain.

Note that the Kelley condition for  $\mathfrak{B}$  implies that  $\mathfrak{B}$  has the  $\sigma$ -finite cc. The reverse implication is not true, as shown by an example of Gaifman [15], or a later example in Argyros in [1].

A measure on a compact  $K$  space is always assumed to be a (countably additive) nonnegative finite Radon measure (a measure  $\mu$  is Radon is  $\mu(M) = \sup\{\mu(F) : F \text{ compact } \subseteq M\}$  for all measurable  $M$ ). If  $K$  is the Stone space of a Boolean algebra  $\mathfrak{B}$  then any finitely additive measure  $\mu$  on  $\mathfrak{B}$  induces in a standard way a countably additive Radon measure  $\hat{\mu}$  on  $K$  which extends  $\mu$  in the sense that  $\hat{\mu}([b]) = \mu(b)$  for the basic clopen set  $[b]$  determined by the element  $b \in \mathfrak{B}$ . The properties of measures that will be defined both for Boolean algebras and compact spaces all satisfy that if  $\mu$  is a finitely additive measure  $\mu$  on a Boolean algebra  $\mathfrak{B}$  satisfying the named property, then the same is true of the induced measure on the Stone space of  $\mathfrak{B}$ . In particular, by a measure on a Boolean algebra we shall mean a finitely additive nonnegative measure. Since we deal only with finite measures we shall for definiteness also assume that all measures in question are probabilities.

**Definition 0.2** (1) A strictly positive measure on Boolean algebra  $\mathfrak{B}$  (a compact space  $K$ ) is a finitely additive (Radon) measure in which every positive element (nonempty open set) has a strictly positive measure.

(2) A measure  $\mu$  on a Boolean algebra  $\mathfrak{B}$  (compact space  $K$ ) is separable if there is a countable  $\mathfrak{A}$  contained in  $\mathfrak{B}$  (the measure algebra of  $\mu$ ) such that for all  $\varepsilon > 0$  and  $b \in \mathfrak{B}$  (in the measure algebra of  $\mu$ ) there is  $a \in \mathfrak{A}$  with  $\mu(a\Delta b) < \varepsilon$ .

(3) A measure on a Boolean algebra (compact space  $K$ )  $\mathfrak{B}$  is nonatomic if for every  $\varepsilon > 0$  there is a finite partition of  $\mathfrak{B}$  ( $K$ ) into elements of measure  $< \varepsilon$ . A Radon measure on a compact space is called continuous if it vanishes at all points.

The terminology from Definition 0.2 is standard. The conditions of nonatomicity and continuity are the same for Radon measures. Also note that the notion of continuity for submeasures, as referred to in the Introduction is different than the notion from Definition 0.2(3). We shall not deal with submeasures in this paper.

Kelley proved in [19] (see also [13], 391 J) that the Kelley condition on a Boolean algebra is necessary and sufficient for the algebra to carry a strictly positive measure. We are interested to find a condition which would necessitate the Boolean algebra to carry a *separable* strictly positive measure. A reasonable candidate for such a condition was proposed by Talagrand in [25], through a notion which we give below and name approximability.

**Definition 0.3** A compact space  $K$  is said to be approximable if there is a sequence  $\langle \mu_n : n < \omega \rangle$  of probability measures on  $K$  such that for every open  $O \subseteq K$  there is  $n$  such that  $\mu_n(O) > 1/2$ . A Boolean algebra  $\mathfrak{B}$  is approximable if its Stone space is approximable.

Any approximable space carries a strictly positive measure, namely the weighted sum of the measures exemplifying approximability. Talagrand's motivation was from

the study of the space of measures,  $M(K)$  on a compact space  $K$ . Approximability of a compact space  $K$  is equivalent to the space  $M(K)$  being weakly separable, and it can also be shown that it is equivalent to  $C(K)$  being isomorphic to a subspace of  $l^\infty$  (see [16]). Talagrand showed that under  $CH$  approximability is not sufficient for the existence of a separable strictly positive measure. A further motivation for the study of this question comes from  $C^*$ -algebras, see [25] for an explanation and further references. The class of approximable Boolean algebras was also studied in [20].

We also recall some standard facts on measures on Boolean algebras which will be useful in the sequel.

**Fact 0.4** 1. *If  $\mu$  is a measure on  $\mathfrak{A}$  and  $\mathfrak{B}$  is some larger algebra then  $\mu$  admits an extension to a measure  $\nu$  on  $\mathfrak{B}$ . Moreover,  $\nu$  can be defined so that  $\inf\{\nu(B \triangle A) : A \in \mathfrak{A}\} = 0$  for every  $B \in \mathfrak{B}$ . In particular, if  $\mu$  is separable on  $\mathfrak{A}$  then  $\mu$  can be extended to a separable measure on any  $\mathfrak{B} \supseteq \mathfrak{A}$ .*

2. *Let  $\mathfrak{C}$  be the measure algebra of  $[0, 1]^\kappa$  with its product measure  $\lambda_\kappa$ , for some  $\kappa > \aleph_0$ . If  $\mu$  is a separable measure on  $\mathfrak{C}$  then  $\mu$  is singular with respect to  $\lambda_\kappa$ ; moreover, for every  $\varepsilon > 0$  there is  $a \in \mathfrak{C}$  such that  $\mu(a) = 0$  while  $\lambda_\kappa(a) \geq 1 - \varepsilon$ . In particular, there is no strictly positive separable measure on  $\mathfrak{C}$ .*

The paper is organised as follows. In the first section we show a  $ZFC$  example of a Boolean algebra which is approximable but does not carry a strictly positive separable measure. The second section presents a specific ccc forcing construction which to every sigma-finite cc Boolean algebra associates a sequence of measures witnessing that the algebra is approximable. A special feature of this sequence is that if the Boolean algebra is atomless then each measure in the sequence is nonatomic. We draw the conclusion that under  $MA + \neg CH$  all atomless ccc Boolean algebras of size  $< \mathfrak{c}$  carry a nonatomic strictly positive measure. We also present a combinatorial characterisation of those Boolean algebras that carry a nonatomic strictly positive measure, given in terms of a chain condition. A corollary of this and the previous theorems above is the equivalence under  $MA + \neg CH$  between the ccc property of an atomless Boolean algebra of size  $< \mathfrak{c}$  and this stronger chain condition.

In section §3 we also present two examples, showing that it is consistent that there is a ccc Boolean algebra of size  $< \mathfrak{c}$  without any strictly positive measure, or that there is a one which carries a strictly positive measure without carrying any separable such measure. Already in §1 we point out examples of atomless ccc Boolean algebras that carry no strictly positive nonatomic measure, while carrying some strictly positive measure. We also recall some known results that might be relevant for further research and give some questions.

# 1 An approximable space with no separable strictly positive measure.

In this section we show that Talagrand's notion of approximability is provably not sufficient for the underlying space (or a Boolean algebra) to carry a separable strictly positive measure. Before embarking on that theorem we shall isolate a property of a compact space that will be used.

**Definition 1.1** *An uncountable separable compact space without isolated points has the unique dense set property (UDSP) if there is a countable dense set  $D \subseteq K$  such that whenever  $H \subseteq K$  is an  $F_\sigma$  set disjoint from  $D$  then  $H$  is nowhere dense.*

In [23] Simon (using a somewhat different terminology) constructed an UDSP space. We shall refer to this space as the *Simon space*. (UDSP spaces were previously known to exist under various set-theoretic axioms, see [4] and [18]). For completeness, at the end of this section we give a somewhat simplified *ZFC* construction of such a space, still based on Simon's ideas. UDSP spaces are a good source of counterexamples because of the following observations.

**Observation 1.2** *Let  $K$  be a space with UDSP. If  $\mu$  is a Radon probability measure on  $K$  and  $\mu(D) = 0$  then  $\mu$  is concentrated on a nowhere dense set. In particular,  $K$  carries no strictly positive continuous Radon measure.*

**Proof.** If  $\mu(D) = 0$  then there are closed sets  $F_n \subseteq K \setminus D$  such that  $\mu(F_n) \geq 1 - 1/n$ . Then for  $H = \bigcup_{n \in \omega} F_n$  we have  $\mu(H) = 1$  so  $\mu$  vanishes on  $K \setminus \overline{H}$ . On the other hand, the choice of  $D$  guarantees that  $H$  is nowhere dense, so  $K \setminus \overline{H}$  is a nonempty open set.

Any continuous measure  $\mu$  on  $K$  has the property that  $\mu(D) = 0$ , so the above argument shows that such a  $\mu$  cannot be strictly positive. ★<sub>1.2</sub>

Observation 1.2 also implies that if  $K$  is a space with UDSP then  $K$  admits a strictly positive measure (a weighted sum of point measures on a countable dense set) but carries no strictly positive homogenous measure, since homogeneous measures are continuous. In the following theorem we start from an UDSP space to obtain an example of an approximable space that admits no separable strictly positive measure.

**Theorem 1.3** *There is an approximable space that admits no separable strictly positive measure.*

**Proof.** Let  $K$  be an UDSP space of weight  $\mathfrak{c}$ , such as the Simon space or the space given at the end of this section, and let  $\mathfrak{A}$  be its algebra of clopen sets. Since  $K$  has no isolated points, we can without loss of generality assume that  $\mathfrak{A}$  is an atomless subalgebra of  $\mathcal{P}(\omega)$  and that  $K$  is a compactification of  $\omega$  with  $D = \omega$  a dense set exemplifying that  $K$  has UDSP.

Hence every  $A \in \mathfrak{A}$  is an infinite subset of  $\omega$ . This property will be important in the proof and it is for this reason that we needed  $K$  not to have any isolated points.

Let  $\lambda$  stand for  $\lambda_{\omega_1}$ , the usual product measure on  $\{0, 1\}^{\omega_1}$  and let  $S$  be the Stone space of the corresponding measure algebra; we denote again by  $\lambda$  the induced measure on the algebra  $\mathfrak{C}$  of the clopen subsets of  $S$ .

We need another Boolean algebra, for which we can take any atomless algebra  $\mathfrak{J}$  on  $\omega$  generated by  $\mathfrak{c}$  many independent sets  $I_\xi \subseteq \omega$  (one can easily find such  $I_\xi$  using the fact that the space  $\{0, 1\}^{\mathfrak{c}}$  is separable).

We now consider all members  $\text{seq} = (\text{seq}(n))_{n < \omega}$  of  ${}^\omega \mathfrak{C}$  such that  $\lim_{n \rightarrow \infty} \lambda(\text{seq}(n)) = 1$ , and denote the collection of such sequences by  $\mathcal{S}$ . Since  $|\mathfrak{A}| = |\mathcal{S}| = \mathfrak{c}$  we can fix a 1-1 enumeration  $\{A_\xi : \xi < \mathfrak{c}\}$  of nonempty elements of  $\mathfrak{A}$  and an enumeration  $\{\text{seq}_\xi : \xi < \mathfrak{c}\}$  of  $\mathcal{S}$ .

If  $\text{seq} \in \mathcal{S}$  and  $A \in \mathfrak{A}$  then we write

$$B(A, \text{seq}) = \bigcup_{n \in A} [\text{seq}(n) \times \{n\}],$$

so  $B(A, \text{seq})$  is a result of distributing elements of  $\text{seq}$  along  $A$ . Let  $\mathfrak{B}$  be the algebra generated in  $S \times \omega \times \omega$  by the sets  $B_\xi = B(A_\xi, \text{seq}_\xi) \times I_\xi$  for  $\xi < \mathfrak{c}$ .

**Claim 1.4** *The algebra  $\mathfrak{B}$  is approximable.*

**Proof of the Claim.** For  $B \in \mathfrak{B}$  and  $n, p < \omega$  let  $B^{(n,p)}$  denote the  $(n, p)$ -section of  $B$ , i.e.

$$B^{(n,p)} = \{s \in S : (s, n, p) \in B\}.$$

For such  $n, p$  we have a naturally defined measure  $\lambda_{(n,p)}$  on  $\mathfrak{B}$  defined by  $\lambda_{(n,p)}(B) = \lambda(B^{(n,p)})$ , so  $\lambda_{(n,p)}$  is a copy of  $\lambda$  put on a given projection of  $\mathfrak{B}$ . We shall show that the measures  $\lambda_{(n,p)}$  for  $n, p \in \omega$  demonstrate the approximability of  $\mathfrak{B}$ .

Note that every nonempty  $B \in \mathfrak{B}$  contains a nonempty set  $B_0$  of the form

$$B_0 = \bigcap_{\xi \in \sigma} [B(A_\xi, \text{seq}_\xi) \times I_\xi] \setminus \bigcup_{\eta \in \tau} [B(A_\eta, \text{seq}_\eta) \times I_\eta],$$

where  $\sigma, \tau$  are some finite subsets of  $\mathfrak{c}$  with  $\sigma \cap \tau = \emptyset$ . Fix such  $B_0$ . It follows that  $\bigcap_{\xi \in \sigma} I_\xi \setminus \bigcup_{\eta \in \tau} I_\eta$  is infinite, so we can pick an element  $p$  in this set. Since  $B_0 \neq \emptyset$  then in particular  $A \stackrel{\text{def}}{=} \bigcap_{\xi \in \sigma} A_\xi \neq \emptyset$  is in  $\mathfrak{A}$ . Now for any  $n \in A$  we have

$$\lambda_{(n,p)}(B) \geq \lambda_{(n,p)}(B_0) \geq \lambda\left(\bigcap_{\xi \in \sigma} \text{seq}_\xi(n)\right),$$

and these values converge to 1 since  $A$  is infinite. This verifies the claim.  $\star_{1.4}$

**Claim 1.5** *The algebra  $\mathfrak{B}$  admits no separable strictly positive measure.*

**Proof of the Claim.** Suppose that  $\nu$  is a separable measure on  $\mathfrak{B}$ . For convenience, consider a larger algebra  $\widehat{\mathfrak{B}}$  generated by all the sets of the form  $B(A, \text{seq}) \times I$ , where  $A \in \mathfrak{A}$ ,  $I \in \mathfrak{I} \cup \{\omega\}$  and  $\text{seq} \in \mathcal{S}$ . Then by Fact 0.4 (1)  $\nu$  has an extension to a separable measure  $\widehat{\nu}$  on  $\widehat{\mathfrak{B}}$ .

Having defined  $\widehat{\nu}$  we can consider its projection onto  $\mathfrak{A}$ , namely define the measure  $\mu$  on  $\mathfrak{A}$  by  $\mu(A) = \widehat{\nu}(S \times A \times \omega)$  for  $A \in \mathfrak{A}$ . We can find a decomposition  $\mu = \mu_a + \mu_c$  such that  $\mu_c^*(\{n\}) = 0$  for every  $n$  while  $\mu_a$  is a purely atomic part, i.e.  $\mu_a(A) = \sum_{n \in A} \mu_a^*(\{n\})$  for every  $A \in \mathfrak{A}$ . Note that by the UDSP property for every nonempty  $A \in \mathfrak{A}$  there is a nonempty  $A' \in \mathfrak{A}$  with  $A' \subseteq A$  and  $\mu_c(A') = 0$ .

We also define measures  $\kappa_n$  for  $n < \omega$  on  $\mathfrak{C}$  by letting

$$\kappa_n(C) = \widehat{\nu}^*(C \times \{n\} \times \omega),$$

for any clopen subset  $C$  of  $S$ . Since  $\widehat{\nu}$  is a separable measure so is each  $\kappa_n$  and therefore each  $\kappa_n$  is singular with respect to  $\lambda$ , i.e. we can find  $C_n \in \mathfrak{C}$  such that  $\lambda(C_n) \geq 1 - 1/n$  while  $\kappa_n(C_n) = 0$  (see Fact 0.4(2)).

We have  $(C_n)_{n \in \omega} = \text{seq}_\xi$  for some  $\xi < \mathfrak{c}$ . We can also find a nonempty  $A_\eta \in \mathfrak{A}$  such that  $A_\eta \subseteq A_\xi$  and  $\mu_c(A_\eta) = 0$  by the remark above. Consider now the set

$$B = [B(A_\xi, \text{seq}_\xi) \times I_\xi] \cap [B(A_\eta, \text{seq}_\eta) \times I_\eta] \in \mathfrak{B},$$

which is easily seen to be nonempty. But

$$\nu(B) \leq \widehat{\nu}(B(A_\eta, \text{seq}_\eta) \times \omega) \leq \mu_c(A_\eta) + \sum_n \kappa_n(\text{seq}_\xi(n)) = 0.$$

We have now checked that no separable measure on  $\mathfrak{B}$  can be strictly positive so the claim is verified.  $\star_{1.5}$

It follows that the Stone space of  $\mathfrak{B}$  the properties required by the Theorem.  $\star_{1.3}$

As promised, we now recall Simon's construction from [23] and enclose a slightly simplified proof of his result. For  $n < \omega$  let  $H_n$  be the set of all nondecreasing functions  $\varphi : n \rightarrow \omega$  and let  $H = \bigcup_{n \in \omega} H_n$ . Given  $\varphi \in H$  and  $g \in {}^\omega\omega$ , write

$$U(\varphi, g) = \{\psi \in H : \varphi \subseteq \psi \ \& \ (\forall i \in \text{dom}(\psi) \setminus \text{dom}(\varphi)) \psi(i) \geq g(i)\}.$$

Let  $\mathfrak{A}$  be the algebra in  $P(H)$  generated by all the sets  $U(\varphi, g)$  as above.

**Lemma 1.6** *If  $\varphi \in A \in \mathfrak{A}$  then there is  $g \in {}^\omega\omega$  such that  $U(\varphi, g) \subseteq A$ .*

**Proof.** It is enough to check this for  $A$  of the form

$$A = \bigcap_{i \leq k} U(\varphi'_i, g'_i) \setminus \bigcup_{j \leq m} U(\varphi''_j, g''_j).$$

Choose  $g$  so that  $g \geq g'_i$  for  $i \leq k$  and  $g(n) > \varphi''_j(n)$  whenever  $n \in \text{dom}(\varphi''_j) \setminus \text{dom}(\varphi)$ . Then one can check that  $U(\varphi, g) \subseteq A$ .  $\star_{1.6}$



**Lemma 1.7** *Let  $\mathcal{V}$  be a cover of  $H$  by elements from  $\mathfrak{A}$ . For every  $n \in \omega$  there is a function  $h : \omega \rightarrow \omega$  such that*

$$W_n(h) := \{\varphi \in H : (\forall k \in \text{dom}(\varphi) \setminus n) \varphi(k) \geq h(k)\}$$

*is covered by finitely many sets from  $\mathcal{V}$ .*

**Proof.** The assertion holds for  $n = 0$ : indeed, by Lemma 1.6,  $\emptyset \in U(\emptyset, h) \subseteq V \in \mathcal{V}$  for some  $h$ , and  $W_0(h) = U(\emptyset, h)$ .

We proceed by induction: given  $n$ , and suppose that  $g$  is such that  $W_n(g)$  is covered by a finite subfamily  $\mathcal{V}'$  of  $\mathcal{V}$ . Let  $m = g(n)$  and let  $G$  be the set of  $\varphi \in H_{n+1}$  with values  $< m$ . For every  $\varphi \in G$  we may find a function  $g_\varphi$  such that  $U(\varphi, g_\varphi)$  is contained in some  $V_\varphi \in \mathcal{V}$ . Finally let  $h$  be the function defined as the maximum of  $g$  and all  $g_\varphi$  for  $\varphi \in G$ . Then  $W_{n+1}(h)$  is covered by  $\mathcal{V}' \cup \{V_\varphi : \varphi \in G\}$ . Indeed, if  $\psi \in W_{n+1}(h) \setminus W_n(g)$  then  $n \in \text{dom}(\psi)$  and  $\psi(n) < g(n) = m$ . Hence  $\varphi = \psi|(n+1) \in G$  (as  $\psi$  is nondecreasing) and  $\psi \in U(\varphi, g_\varphi) \subseteq V_\varphi$ .  $\star_{1.7}$

**Theorem 1.8** *The space  $K = \text{ULT}(\mathfrak{A})$  is an UDSP space, and specifically the set  $H$  under its natural identification as a subset of  $K$  is a countable dense set in  $K$  such that every  $F_\sigma$  subset of  $K$  disjoint from  $H$  is nowhere dense.*

**Proof.** Every  $U(\varphi, g)$  is infinite so by Lemma 1.6 every nonempty  $A \in \mathfrak{A}$  is infinite; it follows easily that  $\mathfrak{A}$  is atomless and hence  $K$  has no isolated points. We identify  $H$  as a subset of  $K$ .

Clearly  $H$  is countable and dense in  $K$ . To check the remaining property of  $H$  it is enough to consider a sequence  $\widehat{\mathcal{V}}_n$ , where

$$\widehat{\mathcal{V}}_n = \{\widehat{V} : V \in \mathcal{V}_n\},$$

and every  $\mathcal{V}_n$  is a cover of  $H$  by elements from  $\mathfrak{A}$ . Fix  $\varphi_0 \in H$  and we shall show that  $\varphi_0$  lies in the interior of  $\bigcap_{n \in \omega} \bigcup \widehat{\mathcal{V}}_n$ .

By Lemma 1.7 there is for every  $n$  a function  $h_n$  such that  $W_n(h_n)$  is covered by a finite subfamily of  $\mathcal{V}_n$ , and we may define  $h : \omega \rightarrow \omega$  as  $h(n) = \max_{i \leq n} h_i(n)$  for  $n < \omega$ . Let  $k = \text{dom}(\varphi_0)$ .

For every  $n \geq k$  we have  $U(\varphi_0, h) \subseteq W_n(h_n)$ , so  $U(\varphi_0, h)$  is covered by a finite number of elements from  $\mathcal{V}_n$ . Hence

$$\widehat{U(\varphi_0, h)} \subseteq \bigcup_{V \in \mathcal{V}_n} \widehat{V},$$

for every  $n \geq k$  and therefore  $\varphi_0 = \emptyset$  lies in the interior of  $\bigcap_{n \geq k} \bigcup \widehat{\mathcal{V}}_n$ . Consequently, it lies in the interior of  $\bigcap_{n < \omega} \bigcup \widehat{\mathcal{V}}_n$ .  $\star_{1.8}$

Simon [23] shows also that there is a whole family of topologies on  $\omega^{<\omega}$  giving spaces with UDSP. For instance, if  $\mathcal{F}$  is any nonprincipal  $P$ -filter in  $P(\omega)$  then such a topology

$\tau(\mathcal{F})$  can be defined by declaring that a set  $U \subseteq \omega^{<\omega}$  open if for every  $s \in U$  the set  $\{n : t \frown \langle n \rangle \in U\}$  is in  $\mathcal{F}$ . It was noticed by Boban Veličković that one can prove that the Stone–Čech compactification of  $\tau(\mathcal{F})$  gives a space with *UDSP* by an argument analogous to the one presented above. For this one can use a description of clopen sets in  $\tau(\mathcal{F})$ , see Błaszczyk & Szymański [4]. It follows that if  $\mathcal{F}$  is a  $P$ -point ultrafilter then we obtain a *UDSP* space which is in addition extremally disconnected (Corollary 13 in [4]).

## 2 A ccc forcing of nonatomic strictly positive measures.

**Theorem 2.1** *For every  $\sigma$ -finite cc Boolean algebra  $\mathfrak{B}$  there is a ccc forcing which makes the algebra approximable (and forces the size of the algebra to be  $\leq \mathfrak{c}$ ). Moreover, if  $\mathfrak{B}$  is atomless, then each measure in the sequence exemplifying approximability is nonatomic - consequently the algebra carries a nonatomic strictly positive measure.*

The point of this theorem is the conclusion in the second sentence, because the conclusion from the first sentence already follows from the known theorems. Namely, if  $\mathfrak{B}$  satisfies that its every power  $\mathfrak{B}^n$  is ccc, which is clearly the case of the  $\sigma$ -finite cc Boolean algebras, then by forcing with  $\mathfrak{B}^\omega$  with finite support one can make  $\mathfrak{B}$   $\sigma$ -centered. This means that the Stone space of the algebra is separable, and hence it supports a separable strictly positive measure, namely a weighted sum of the point masses of points in the countable dense set. We include a simple proof of the instance of this most relevant to us, (see [14], 43F (b) for more discussion and references):

**Fact 2.2** *Suppose that  $MA + \neg CH$  holds. Then every ccc compact space of  $\pi$ -weight  $< \mathfrak{c}$  is separable.*

**Proof.** Let  $K$  be as in the assumptions and let  $P$  be a  $\pi$ -base of  $K$  of cardinality  $< \mathfrak{c}$ . Under  $MA + \neg CH$ ,  $K^\omega$  is also ccc and we work in this space. Let for given  $p \in P$  the family  $\mathcal{U}(p)$  consist of basic open rectangles in  $K^\omega$  with at least one side equal  $p$ .

Then the union of  $\mathcal{U}(p)$  is dense open in  $K^\omega$ . Applying  $MA$ , there is  $x \in K^\omega$  which is in  $\bigcup \mathcal{U}(p)$  for every  $p \in P$ . Let  $x = (x_n)_n$ . We claim that  $\{x_n : n < \omega\}$  is a dense set in  $K$ . Namely, if  $U$  is open nonempty in  $K$  then there is  $p \in P$  with  $p \subseteq U$ . But  $x \in \bigcup \mathcal{U}(p)$  implies  $x_n \in p$ , for some  $n$ , hence  $x_n \in U$ .  $\star_{2.2}$

Using this fact and the point mass measures we obtain that if  $MA + \neg CH$  holds then any ccc Boolean algebra of size  $< \mathfrak{c}$  carries a separable strictly positive measure (and is hence certainly approximable). However, since this measure is induced by a weighted sum of point masses on the Stone space, the measure is clearly not nonatomic. The question is if we can also obtain such a measure which is nonatomic. The conclusion of our theorem is that this is indeed the case if the Boolean algebra we start is atomless. Approximability is simply used as a tool.

After giving the proof of the theorem we shall spell out its corollary under  $MA+\neg CH$  and further discuss spaces of weight  $< \mathfrak{c}$ . Let us now carry on to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let  $\mathfrak{B}$  satisfy the assumptions of the theorem and let  $\mathfrak{B} = \bigcup_{n < \omega} \mathcal{B}_n$  be such that no  $\mathcal{B}_n$  has an infinite antichain. We shall define a forcing notion  $\mathbb{P}$  as required.  $\mathbb{P}$  is defined naturally, so its conditions are of the form

$$p = (\mathfrak{A}^p, F^p, \langle \mu_n^p : n \in F^p \rangle)$$

where  $\mathfrak{A}^p$  is a finite subalgebra of  $\mathfrak{B}$ ,  $F^p$  is a finite subset of  $\omega$ , and for every  $n \in F^p$  we have  $\mu_n^p$  which is a finitely additive probability measure on  $\mathfrak{A}^p$  taking *rational* values. The extension is also defined in a natural way, so  $p \leq q$  ( $q$  is stronger) if  $\mathfrak{A}^p \subseteq \mathfrak{A}^q$ ,  $F^p \subseteq F^q$  and for every  $n \in F^p$  we have that  $\mu_n^q \upharpoonright \mathfrak{A}^p = \mu_n^p$ .

It will be easily checked that the forcing makes  $\mathfrak{B}$  approximable, and the main point will be to verify that the forcing is in fact ccc. For this we shall use the following amalgamation theorem due to Strassen [24] (see also [14] 453A, 453C, 453D), which also can be used to verify the various statements of the following Claim 2.4.

**Amalgamation Theorem 2.3** (*Strassen*) *Suppose that we have two Boolean subalgebras  $\mathfrak{A}_0$  and  $\mathfrak{A}_1$  of a Boolean algebra  $\mathfrak{A}$  with measures  $\mu_0$  and  $\mu_1$  respectively, satisfying that  $\mu_0 \upharpoonright (\mathfrak{A}_0 \cap \mathfrak{A}_1) = \mu_1 \upharpoonright (\mathfrak{A}_0 \cap \mathfrak{A}_1)$ . A necessary and sufficient condition for there to exist a measure  $\mu$  on  $\langle \mathfrak{A}_0 \cup \mathfrak{A}_1 \rangle_{\mathfrak{A}}$  extending both  $\mu_0$  and  $\mu_1$  is that for every  $l \in \{0, 1\}$ ,  $a \in \mathfrak{A}_l$  and  $b \in \mathfrak{A}_{1-l}$ , if  $a \subseteq b$  then  $\mu_l(a) \leq \mu_{1-l}(b)$ .*

**Claim 2.4**  $\mathbb{P}$  forces  $\mathfrak{B}$  to be approximable.

**Proof of the Claim.** Let  $G$  be  $\mathbb{P}$ -generic and for each  $n$  let  $\mu_n = \bigcup \{ \mu_n^p : p \in G \}$ . Applying Theorem 2.3, it easily follows by genericity that each  $\mu_n$  is a finitely additive probability measure on  $\mathfrak{B}$ . If  $b \in \mathfrak{B}^+$  then the set

$$\mathcal{D}_b = \{ p \in \mathbb{P} : b \in \mathfrak{A}^p \ \& \ \mu_n^p(b) > 1/2 \text{ for some } n \in F^p \}$$

is dense, because if  $b \in \mathfrak{B}^+$  is given, we can let  $\mathfrak{A}^q$  be  $\langle \mathfrak{A}^p \cup \{b\} \rangle_{\mathfrak{B}}$  and choose  $n \notin F^p$  to define  $\mu_n^q$  on  $\mathfrak{B}^q$  so that  $\mu_n^q(b) > 1/2$ .  $\star_{2.4}$

**Main Claim 2.5**  $\mathbb{P}$  is ccc.

**Proof of the Claim.** Suppose that we are given  $\aleph_1$  many distinct conditions  $\{ p_\alpha : \alpha < \omega_1 \}$  in  $\mathbb{P}$ . We shall denote  $\mathfrak{A}^{p_\alpha}$  by  $\mathfrak{A}_\alpha$ ,  $F^{p_\alpha}$  by  $F_\alpha$  and  $\mu_i^{p_\alpha}$  by  $\mu_i^\alpha$ . By passing to a subset if necessary we can assume that all  $F_\alpha$  are the same set  $F$  and that  $\mathfrak{A}_\alpha$ 's form a  $\Delta$ -system with root  $\mathfrak{A}^*$ . We may also assume that for every  $o \in F$  the restriction  $\mu_o^\alpha \upharpoonright \mathfrak{A}^*$  is fixed, since the measures only assume rational values.

By further trimming if necessary, we may assume that there is a fixed number  $m^* \geq 1$  such that each  $\mathfrak{A}_\alpha$  is generated over  $\mathfrak{A}^*$  by  $m^*$  many additional elements

$A_\alpha = \{a_0^\alpha, \dots, a_{m^*-1}^\alpha\}$ . Since the Boolean algebras in question are finite and the measures take rational values, we may assume that for every  $o \in F$  and Boolean formula  $\varphi(x_0, \dots, x_{m^*-1}; \mathfrak{A}^*)$  with parameters in  $\mathfrak{A}^*$ , denoting  $b_\alpha^\varphi = \varphi[a_0^\alpha, \dots, a_{m^*-1}^\alpha; \mathfrak{A}^*]$  for  $\alpha < \omega_1$ , the measure  $\mu_o^\alpha(b_\alpha^\varphi)$  does not depend on  $\alpha$ . We may also assume that for each such  $\varphi$  there is a fixed  $n = n^\varphi$  such that all  $b_\alpha^\varphi$  belong to  $\mathcal{B}_n$ . Using Ramsey theorem we can assume that for the first  $\omega$  many  $\alpha$ , the sequences  $\langle a_0^\alpha, \dots, a_{m^*-1}^\alpha \rangle$  are 2-indiscernible over  $\mathfrak{A}^*$ , so for any  $\gamma < \delta < \omega$  and any Boolean formula  $\varphi(x_0, \dots, x_{2m^*-1}; \mathfrak{A}^*)$  with parameters in  $\mathfrak{A}^*$ , the truth value of  $\varphi(a_0^\gamma, \dots, a_{m^*-1}^\gamma, a_0^\delta, \dots, a_{m^*-1}^\delta; \mathfrak{A}^*)$  does not depend on the actual values of  $\gamma < \delta$ .

We now show that for every  $\alpha, \beta < \omega$  the conditions  $p_\alpha$  and  $p_\beta$  are compatible. Let us fix such  $\alpha, \beta$ . We need to show that for every  $o \in F$  there is a measure  $\mu_o$  on  $\langle \mathfrak{A}_\alpha \cup \mathfrak{A}_\beta \rangle_{\mathfrak{B}}$  which extends  $\mu_o^\alpha \cup \mu_o^\beta$ . It suffices to show this for one  $o$  at a time, so we can fix such  $o$  and for simplicity in notation we shall write  $\mu_\alpha$  for  $\mu_o^\alpha$ . We need to verify the amalgamation condition from the Amalgamation Theorem. It will be more convenient to use the set theoretic notation, and we shall use the fact that the measures are finitely additive.

Let  $\{f_0, \dots, f_{2m^*-1}\}$  enumerate  ${}^{m^*}2$ . It is well known (see [22]) that every element of  $\mathfrak{A}_\alpha$  is obtained as the disjoint union  $\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\alpha^i)^{f(i)} \cap x_f]$  for some  $\{x_{f_0}, \dots, x_{f_{2m^*-1}}\}$  in  $\mathfrak{A}^*$ , and similarly for  $\mathfrak{A}_\beta$ . Suppose that for some  $\{x_f, y_f : f \in {}^{m^*}2\}$  in  $\mathfrak{A}^*$  we have

$$\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\alpha^i)^{f(i)} \cap x_f] \subseteq \bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\beta^i)^{f(i)} \cap y_f].$$

Suppose for a contradiction that

$$\mu_\alpha\left(\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\alpha^i)^{f(i)} \cap x_f]\right) > \mu_\beta\left(\bigcup_{h \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\beta^i)^{h(i)} \cap y_h]\right).$$

By one of our assumptions we have

$$\mu_\alpha\left(\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\alpha^i)^{f(i)} \cap x_f]\right) = \mu_\beta\left(\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\beta^i)^{f(i)} \cap x_f]\right).$$

Substituting and simplifying we obtain that it must be the case that

$$\mu_\beta\left(\bigcup_{f \in {}^{m^*}2} [\bigcap_{i < m^*} (a_\beta^i)^{f(i)} \cap x_f \cap \bigcap_{h \in {}^{m^*}2} y_h^c]\right) > 0,$$

so there must be  $f \in {}^{m^*}2$  such that

$$\bigcap_{i < m^*} (a_\beta^i)^{f(i)} \cap x_f \cap \bigcap_{h \in {}^{m^*}2} y_h^c \neq \emptyset.$$

This implies that for every  $\gamma < \omega_1$  we have  $\bigcap_{i < m^*} (a_\gamma^i)^{f(i)} \cap x_f \cap \bigcap_{h \in {}^{m^*}2} y_h^c \neq \emptyset$ , and applying the fact that all these elements of  $\mathfrak{B}$  come from the same subset of  $\mathfrak{B}$  in

which there are no infinite antichains we obtain that there are  $\gamma < \delta < \omega$  such that  $\bigcap_{i < m^*} [(a_\gamma^i)^{f(i)} \cap (a_\delta^i)^{f(i)}] \cap x_f \cap \bigcap_{h \in m^* 2} y_h^c \neq \emptyset$ . By indiscernibility we have that

$$\bigcap_{i < m^*} [(a_\alpha^i)^{f(i)} \cap (a_\beta^i)^{f(i)}] \cap x_f \cap \bigcap_{h \in m^* 2} y_h^c \neq \emptyset.$$

However,

$$\bigcap_{i < m^*} [(a_\alpha^i)^{f(i)} \cap x_f \cap (a_\beta^i)^{f(i)}] \subseteq \bigcup_{h \in m^* 2} [\bigcap_{j < m^*} (a_\beta^j)^{h(j)} \cap y_h] \cap \bigcap_{i < m^*} (a_\beta^i)^{f(i)}.$$

Noticing that for  $f \neq h$  we have  $\bigcap_{i < m^*} (a_\beta^i)^{f(i)} \cap \bigcap_{j < m^*} (a_\beta^j)^{h(j)} = \emptyset$ , we obtain that the right-hand side is simply  $\bigcap_{j < m^*} (a_\beta^j)^{f(j)} \cap y_f$ , which is disjoint from  $\bigcap_{j < m^*} y_h^c$ , a contradiction.  $\star_{2.5}$

Finally we shall note that all the measures on the sequence exemplifying approximability are nonatomic, hence certainly their weighted sum is nonatomic as well.

**Claim 2.6** *Let  $\mathbb{P}$ ,  $\mathfrak{B}$  be as in above and further suppose that  $\mathfrak{B}$  is atomless. Let  $\langle \mu_n : n < \omega \rangle$  in  $V^\mathbb{P}$  be the sequence of measures obtained by forcing with  $\mathbb{P}$ . Then for each  $n$  the measure is nonatomic.*

**Proof.** Let  $m, n < \omega$ . We shall show that the set of conditions in  $\mathbb{P}$  which force that there is a partition of unity in  $\mathfrak{B}$  in which each element has  $\mu_n$ -measure  $< 1/m$ , is dense in  $\mathbb{P}$ .

Given  $p \in \mathbb{P}$ . By defining  $\mu_n^p$  trivially if necessary we can assume  $n \in F^p$ . Let  $\{c_0, c_1, \dots, c_{k-1}\}$  be the atoms of  $\mathfrak{A}^p$ , which exist as  $\mathfrak{A}^p$  is finite. Note that these atoms form a partition of unity in  $\mathfrak{A}^p$ . It suffices to define by induction on  $i \leq k$  a sequence  $q_0 = p \leq q_1 \leq \dots \leq q_k$  of extensions of  $p$  such that in each  $\mathfrak{A}^{q_{i+1}}$  there is a partition of  $c_i$  into pieces of  $\mu_n$ -measure  $< 1/m$ . Given  $q_i$ , let  $\{d_j^i : j < j_i\}$  be the atoms of  $\mathfrak{A}^{q_i}$  which are contained in  $c_i$ . Since  $\mathfrak{B}$  is atomless we can for each  $j$  find a partition of  $d_j^i$  into at least  $m+1$  disjoint pieces  $e_0^{i,j}, \dots, e_{m+1}^{i,j}$ . Let  $\mathfrak{A}^i$  be the algebra with the largest element  $c_i$  and generated by all  $e_0^{i,j}, \dots, e_{m+1}^{i,j}$  for  $j < j_i$ . We can define a measure  $\mu^i$  on  $\mathfrak{A}^i$  with  $\mu^i(d_j^i) = \mu_n^{q_i}(d_j^i)$  and  $\mu(e_l^{i,j}) < 1/m$  for every  $l, j$ . By the Amalgamation Lemma and the fact that  $\mathfrak{A}^i \cap \mathfrak{A}^{q_i}$  is the algebra generated by all  $d_j^i$  for  $j < j_i$ , we can find a measure  $\mu_n^i \supseteq \mu_n^{q_i} \cup \mu^i$  defined on  $\langle \mathfrak{A} \cup \mathfrak{A}^{q_i} \rangle$ , and hence we can extend  $q_i$  to a condition in  $q_{i+1}$  as required. The conclusion follows by considering  $q_k$   $\star_{2.6}$

$\star_{2.1}$

**Note 2.7** *If  $\mathfrak{B}$  is approximable then  $C(K)$  embeds into  $l^\infty$  (see [25]), where  $K$  is the Stone space of  $\mathfrak{B}$ . Hence we would expect  $\mathfrak{B}$  to have the size of at most the continuum in  $V^\mathbb{P}$ . This is exactly what happens because one can easily see that  $\mathbb{P}$  adds  $|\mathfrak{B}|$  many reals,*

$$\langle \mu_n(b) : n < \omega \rangle \text{ for } b \in \mathfrak{B}.$$

With the notation of Theorem 2.1 we note that  $\mathcal{B}$  has the Kelley property in  $V^{\mathbb{P}}$  (as it is approximable), and hence it carries a strictly positive measure. In  $V^{\mathbb{P}}$  the size of the algebra is  $\leq \mathfrak{c}$ , so the Radon measure induced on the Stone space of  $\mathcal{B}$  has Maharam dimension at most  $\mathfrak{c}$ . By a result of Dow and Steprans in [7] this means that in  $V^{\mathbb{P}}$  the algebra  $\mathcal{B}$  must be  $\sigma$ - $n$ -linked for every  $n$  (see section 3 for a discussion).

We can spell out the meaning of Theorem 2.1 in the context of  $MA + \neg CH$ :

**Corollary 2.8** *If  $MA + \neg CH$  holds then every ccc topological 0-dimensional space of weight  $< \mathfrak{c}$  and no isolated points supports a strictly positive continuous measure. Similarly, every ccc atomless Boolean algebra of size  $< \mathfrak{c}$  carries a strictly positive nonatomic measure.*

**Proof.** For the first statement, let  $K$  be such a space, so it is the Stone space of a ccc Boolean algebra  $\mathcal{B}$  of size  $\mathfrak{c}$  which is atomless. Since by Fact 2.2  $K$  is separable it certainly supports some strictly positive measure, namely a weighted sum of point measures. In particular  $\mathcal{B}$  satisfies Kelley's condition and hence by Theorem 2.1 has a sequence  $\langle \mu_n : n < \omega \rangle$  exemplifying its approximability, and obtained by forcing with  $\mathbb{P}$ . By Claim 2.6 a weighted sum of these measures will be a measure as required.

The second statement is proved similarly and even more directly.  $\star_{2.8}$

A theorem by Mägerl and Namioka (see Theorem 3.4) shows that approximability of a Boolean algebra is a chain condition. It remains unclear if there is an analogous chain condition characterising algebras with strictly positive separable measures. We note, however, that algebras with strictly positive nonatomic measures can be characterised as follows.

**Theorem 2.9** *A Boolean algebra  $\mathfrak{B}$  carries a strictly positive nonatomic measure iff there is a decomposition  $\mathfrak{B} \setminus \{0\} = \bigcup_{n < \omega} \mathfrak{B}_n$ , where for each  $n$  we have*

- (i)  $\mathfrak{B}_n \subseteq \mathfrak{B}_{n+1}$ ;
- (ii)  $\text{int}(\mathfrak{B}_n) \geq 2^{-n}$ ;
- (iii) if  $a \in \mathfrak{B}_n$  then there are disjoint  $b, c \in \mathfrak{B}_{n+1}$  with  $b \cup c \leq a$ .

We shall take for granted the following fact, appearing as part of Kelley's proof in [19] and taken in this form from [13], Proposition 391 I.

**Fact 2.10** *Let  $\mathfrak{A}$  be a Boolean algebra and  $A \subseteq \mathfrak{A} \setminus \{0\}$  nonempty. Then*

$$\text{int}(A) = \max_{\nu} \inf_{a \in A} \nu(a),$$

where max is taken over all probability (finitely additive) measures on  $\mathfrak{A}$ .

**Proof of Theorem 2.9.** If  $\mu$  is a strictly positive nonatomic measure on  $\mathfrak{B}$  then we take  $\mathfrak{B}_n = \{b \in \mathfrak{B} : \mu(b) > 2^{-n}\}$ . Then (i) is obvious and (ii) is easily seen to hold.

Condition (iii) follows from nonatomicity. It is well-known that if  $\nu$  is a nonatomic Radon measure (on a compact space  $K$ ) then  $\nu$  has the following Darboux property: for every Borel set  $B \subseteq K$  and  $0 < r < \nu(B)$  there is a compact set  $F \subseteq B$  with  $\nu(F) = r$ . We can apply this remark to  $\nu = \widehat{\mu}$ , the Radon measure corresponding to  $\mu$  on the Stone space  $K$  of  $\mathfrak{B}$ . Since  $\mu$  is nonatomic, so is  $\nu$ . Suppose that  $a \in \mathfrak{B}_n$ , so  $\widehat{a}$  is a nonempty clopen set in  $K$  with  $\nu(\widehat{a}) > 2^{-n}$  and therefore there are disjoint compact sets  $F_1, F_2 \subseteq \widehat{a}$  of measure  $> 2^{-n-1}$ . Now there are disjoint  $b, c \in \mathfrak{B}$  such that  $b, c \leq a$  and  $F_1 \subseteq \widehat{b}, F_2 \subseteq \widehat{c}$ , and this verifies (iii).

If there is a decomposition of  $\mathfrak{B}$  satisfying (i)–(iii) then by Fact 2.10 for each  $n$  we can define a probability measure  $\mu_n$  on  $\mathfrak{B}$  such that for all  $b \in \mathfrak{B}_n$  we have  $\mu_n(b) \geq 2^{-n}$ . We let  $\mu$  be any cluster point of the sequence  $(\mu_n)_n$ . It is easily seen that  $\mu$  is a probability measure on  $\mathfrak{B}$ .

Note that if  $a \in \mathfrak{B}_n$  then  $\mu(a) \geq 2^{-n}$ . Indeed, by induction on  $k$  it easily follows that for all  $k \geq n$  there are  $2^{k-n}$  pairwise disjoint elements in  $\mathfrak{B}_k$  contained in  $a$ . Hence for such  $n, k$  we have  $\mu_k(a) \geq 2^{k-n} \cdot 2^{-k} = 2^{-n}$ . Consequently,  $\mu(a) \geq 2^{-n} > 0$ .

Hence  $\mu$  is strictly positive. We can wlog assume that  $1 \in \mathfrak{B}_0$ , and apply the previous remark to  $a = 1$ : for every  $n$  there are pairwise disjoint  $b_i \in \mathfrak{B}_n, 0 \leq i \leq 2^n - 1$ . Then  $\mu(b_i) \geq 2^{-n}$  so necessarily  $\mu(B_i) = 2^{-n}$  and this shows that  $\mu$  is nonatomic.  $\star_{2.9}$

Theorem 2.9 allows us to state the following

**Corollary 2.11** *Assume  $MA + \neg CH$ . Then for atomless Boolean algebras  $\mathfrak{B}$  of size  $< \mathfrak{c}$ , the following are equivalent*

- (i)  $\mathfrak{B}$  is ccc, and
- (ii)  $\mathfrak{B}$  satisfies the chain condition from Theorem 2.9.

Finally we remark that the method of the proof of Theorem 2.1 can be seen as a template for obtaining various measures on a Boolean algebra with different properties. For the Corollary 2.8 we could have used a subcase of the method in which we would have only forced one measure and required it to be strictly positive and nonatomic.

### 3 Boolean algebras of small size and some combinatorial conditions.

Corollary 2.8 suggests a few more questions about Boolean algebras of size  $< \mathfrak{c}$ , which we shall consider in this section. We shall first show that Corollary 2.8 cannot be proved in  $ZFC$ . In the original version of the paper we noticed that by a modification of a classical Gaifman space, see e.g. [6] Theorem 6.23 or [2], one can define a Boolean algebra of size  $\text{non}(\mathcal{M})$ , which is ccc (in fact satisfies Knaster's condition) but carries no strictly

positive measure. The referee remarked that a construction due to Todorčević [28], Theorem 8.4, gives an analogous algebra of cardinality the smaller cardinal invariant  $\text{add}(\mathcal{N})$ , the additivity of the Lebesgue measure. We give a sketch of the argument.

**Theorem 3.1** (*Todorčević*) *There is a Boolean algebra  $\mathfrak{A}$  such that*

(i)  $|\mathfrak{A}| = \text{add}(\mathcal{N})$ ,

(ii)  $\mathfrak{A}$  is ccc but not  $\sigma$ -centred,

(iii)  $\mathfrak{A}$  is generated by a subfamily  $\mathcal{G}$  with the property that if  $a, b \in \mathcal{G}$  then  $a \leq b$ ,  $b \leq a$  or  $a \cdot b = 0$ .

*Consequently, there is no strictly positive measure on  $\mathfrak{A}$ .*

**Proof.** In [28] Theorem 8.4 there is a construction of a Boolean algebra  $\mathfrak{B}$  given by two sets of generators. Taking only the first kind of generators  $T_a$  from that construction we obtain a subalgebra  $\mathfrak{A}$  of  $\mathfrak{B}$  satisfying the properties (i)-(iii) above. We shall only check that (ii) and (iii) imply that  $\mathfrak{A}$  carries no strictly positive measure, because this fact is not mentioned explicitly in [28].

**Lemma 3.2** *Suppose that an algebra  $\mathfrak{A}$  is generated by a subfamily  $\mathcal{G}$  such that if  $a, b \in \mathcal{G}$  then  $a \leq b$ ,  $b \leq a$  or  $a \cdot b = 0$ , and that  $\mathfrak{A}$  is not  $\sigma$ -centred. Then  $\mathfrak{A}$  carries no strictly positive measure.*

**Proof.** Assume that  $\mu$  is a strictly positive measure on  $\mathfrak{A}$ . Fix  $r > 0$  and consider  $G_r = \{g \in \mathcal{G} : \mu(g) > r\}$ . Let  $L_0$  be a maximal linearly ordered part of  $G_r$ ; denote  $r_0 = \inf\{\mu(a) : a \in L_0\}$  and take  $a_0 \in L_0$  such that  $\mu(a_0) < r_0 + r/2$ .

Then  $L_1$  be a maximal linearly ordered part of  $G \setminus L_0$ ; define  $r_1$  and  $a_1$  as above, etc.

The point is that for  $k \neq l$  the elements  $a_k$  and  $a_l$  are disjoint. Let us show this on the example of  $a_0$  and  $a_1$ : indeed, by maximality of  $L_0$  there are  $x_0 \in L_0$  and  $x_1 \in L_1$  such that  $x_0 x_1 = 0$ . Suppose that  $a_0$  and  $a_1$  are not disjoint. We have  $\mu(a_0 - x_0) < r/2$ ,  $\mu(x_1) \geq r$ , so  $x_1$  cannot be below  $a_0$ . If  $x_1 \leq a_1$  this gives that  $a_1$  is not below  $a_0$ . If  $a_1 \leq x_1$  and  $a_1 \leq a_0$  then  $a_1 \leq x_1 \cdot a_0 \leq a_0 \setminus x_0$ , which is a contradiction. Therefore since  $x_1$  and  $a_1$  are both from  $L_1$  we conclude that  $a_1$  cannot be below  $a_0$ .

If  $a_0 \leq a_1$  and  $a_1 \leq x_1$  we get a contradiction with  $x_0 \cdot x_1 = 0$ . If  $a_0 \leq a_1$  and  $x_1 \leq a_1$ , since  $\mu(a_1) < r_1 + r/2$  we have  $\mu(a_1 \setminus x_1) < r/2$ , yet  $a_0 \cdot x_0 \leq a_1 \setminus x_1$ , a contradiction.

It is now clear that the process of defining  $L_n$ 's will stop after at most  $1/r$  steps. Therefore  $\mathcal{G}_r$  is finitely centred and thus  $\mathcal{G}$  (and consequently  $\mathfrak{A}$ ) is  $\sigma$ -centred, a contradiction.  $\star_{3.2}$

$\star_{3.1}$

Next we show an example of small (i.e. of size  $< \mathfrak{c}$ ) Boolean algebra admitting a strictly positive measure but no separable strictly positive measure. Let  $\lambda_\kappa$  be the



usual product measure on  $\{0, 1\}^\kappa$  and let  $\mathcal{N}_\kappa$  be the corresponding ideal of null sets. We consider the measure algebra  $\mathfrak{A}$  of  $\lambda_{\omega_1}$  and its Stone space  $S = \text{ULT}(\mathfrak{A})$ ; let  $\nu$  be the Radon measure on  $S$  induced by  $\lambda_{\omega_1}$ . Recall that  $\text{cof}(\mathcal{N}_\omega) = \text{cof}(\mathcal{N}_{\omega_1})$  agrees with the cofinality of the ideal of  $\nu$ -null sets, see [12]. Also recall that it is consistent that  $\text{cof}(\mathcal{N}_\omega) = \omega_1 < \mathfrak{c}$ ; this is so in the Sacks model, see e.g. [5].

**Theorem 3.3** *Assuming  $\text{cof}(\mathcal{N}_\omega) = \omega_1$  there is a Boolean algebra  $\mathfrak{B}$  of cardinality  $\omega_1$  such that  $\mathfrak{B}$  has a strictly positive measure but carries no strictly positive separable measure.*

**Proof.** By our assumption and the remarks preceding the Theorem we can find a family  $\{Z_\xi : \xi < \omega_1\}$  of closed subsets of  $S$  with  $\lambda(Z_\xi) = 0$ , which is cofinal for the ideal of  $\nu$ -null sets. We can moreover assume that every  $Z_\xi$  is a  $G_\delta$  so we can for every  $\xi < \omega_1$  fix a decreasing sequence  $(a_n^\xi)_{n \in \omega}$  in  $\mathfrak{A}$  such that  $Z_\xi = \bigcap_{n \in \omega} \widehat{a}_n^\xi$ .

Note that for every  $\eta < \omega_1$  there is  $0 \neq b_\eta \in \mathfrak{A}$  such that whenever  $\xi < \eta$  then  $b_\eta \cdot a_n^\xi = 0$  for  $n$  large enough (indeed we have only countably many sequences on which the measure tends to 0).

Let  $\mathfrak{B}$  be the algebra generated by all  $b_\eta$ ,  $\eta < \omega_1$ . Then  $\mathfrak{B}$  is a subalgebra of  $\mathfrak{A}$  of size  $\omega_1$  and clearly  $\mathfrak{B}$  has a strictly positive measure.

Consider any separable measure  $\mu_0$  on  $\mathfrak{B}$ . Then  $\mu_0$  can be extended to a separable measure  $\mu$  on  $\mathfrak{A}$  (see Fact 0.4). The measure  $\mu$ , as a measure on  $S$ , is concentrated on some set  $\bigcup_{n \in \omega} Z_n$  where  $Z_n$  are closed and  $\lambda(Z_n) = 0$  for every  $n$ . We have  $Z_n \subseteq Z_{\xi_n}$  for some  $\xi_n$ ; take any  $\eta$  with  $\omega_1 > \eta > \xi_n$  for every  $n$ . We have

$$\widehat{b}_\eta \cap \bigcup_{n \in \omega} Z_n = \emptyset,$$

which gives  $\mu(b_\eta) = 0$  and hence  $\mu_0(b_\eta) = 0$ , i.e.  $\mu_0$  is not strictly positive on  $\mathfrak{B}$ .  $\star_{3.3}$

For completeness, we mention now a couple of known results about decompositions of Boolean algebras. Theorem 2.3 of [20] gives a combinatorial characterisation of approximable Boolean algebras. The proof uses the equivalence between approximability of a Boolean algebra  $\mathfrak{B}$  and the weak\* separability of the space  $M_1^+(K)$ , where  $K$  is the Stone space of  $\mathfrak{B}$ , and is phrased in terms of  $\pi$ -bases of compact Hausdorff spaces. As the direct argument in the language used here is very simple we include it for convenience. It is conceivable that adding some properties to this characterisation would indeed give a characterisation of Boolean algebras that carry a separable strictly positive measure—we clearly have not been able to do this.

**Theorem 3.4** (Mägerl-Namioka) *A Boolean algebra  $\mathfrak{B}$  is approximable iff for every  $\varepsilon > 0$  (equivalently: for some  $\varepsilon \in (0, 1)$ ) there is a decomposition  $\mathfrak{B} \setminus \{0\} = \bigcup_{n < \omega} \mathfrak{B}_n^\varepsilon$ , where for each  $n$  we have  $\text{int}(\mathfrak{B}_n^\varepsilon) \geq 1 - \varepsilon$ .*

**Proof.** In the forward direction, suppose that  $\langle \mu_n : n < \omega \rangle$  is a sequence of measures exemplifying the approximability of a Boolean algebra  $\mathfrak{B}$ . Given  $\varepsilon > 0$ . Let  $\mathfrak{B}_n^\varepsilon = \{b \in$

$\mathfrak{B} : \mu_n(b) > 1 - \varepsilon\}$ . Since  $\mu_n$  is a measure on  $\mathfrak{B}$  such that  $\mu_n(b) > 1 - \varepsilon$  for all  $b \in \mathfrak{B}_n^\varepsilon$  we have by Fact 2.10 that  $\text{int}(\mathfrak{B}_n^\varepsilon) \geq 1 - \varepsilon$ . It follows from the choice of  $\mu_n$ 's that  $\mathfrak{B} \setminus \{0\} = \bigcup_{n < \omega} \mathfrak{B}_n^\varepsilon$ .

In the other direction let us consider for each  $m \geq 1$  the decomposition  $\mathfrak{B} \setminus \{0\} = \bigcup_{n < \omega} \mathfrak{B}_n^{1/m}$ . Using the choice of these sets and Fact 2.10 we can define a measure  $\mu_n^m$  on  $\mathfrak{B}$  such that for all  $b \in \mathfrak{B}_n^m$  we have  $\mu_n^m(b) \geq 1 - 1/m$ . Reenumerating  $\langle \mu_n^m : n < \omega, 1 \leq m \rangle$  as  $\langle \mu_n : n < \omega \rangle$  we obtain the measures that exemplify that  $\mathfrak{B}$  is approximable.  $\star_{3.4}$

We note that Dow and Steprans in [7] obtain a combinatorial criterion that distinguishes measure algebras of type  $\leq \mathfrak{c}$  for larger ones. Namely, they prove that the measure algebra on  $2^\kappa$  for  $\kappa \leq \mathfrak{c}$  is  $\sigma - n$  linked for each  $n < \omega$ , and it is not  $\sigma - n$  linked for  $\kappa > \mathfrak{c}$ .

We finish by mentioning a question of a different nature. It is clear that every approximable Boolean algebra carries a strictly positive measure, as such a measure can be obtained as a weighted sum of the measures on the sequence exemplifying approximability. The notion of approximability can be easily generalised to higher dimensions:

**Definition 3.5** *A compact space  $K$  is said to be  $\kappa$ -approximable if there is a sequence  $\langle \mu_\alpha : \alpha < \kappa \rangle$  of probability measures on  $K$  such that for every open  $O \subseteq K$  there is  $\alpha$  such that  $\mu_\alpha(O) > 1/2$ . A Boolean algebra  $\mathfrak{B}$  is  $\kappa$ -approximable if its Stone space is  $\kappa$ -approximable.*

Clearly  $K$  is  $\kappa$ -approximable iff  $C(K)$  embeds into  $l^\infty(\kappa)$ . There does not seem to be anything in  $\kappa$ -approximability that guarantees the existence of a strictly positive measure. Also note that  $\kappa = \aleph_0$  is rather special in that every separable compact space is the support of a separable measure, but this fact need not generalise to  $\kappa > \aleph_0$ . Hence we can ask:

**Question 3.6** *Suppose that a Boolean algebra satisfies the Kelley property and is  $\kappa$ -approximable. Is there necessarily a strictly positive measure on  $\mathfrak{B}$  of measure density  $\kappa$ ?*

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