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The simplex method

Notes for the lecture

Mathematical programming and optimization,

to be held in the Spring semester 2021. This part closely follow the book *Introduction to linear optimization* by D. Bertsimas and J. Tsitsiklis.

Notation and terminology; basic facts

For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$ etc, we write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = \langle x, y \rangle,$$

$$x \leqslant y \iff (\forall i \leqslant n) x_i \leqslant y_i,$$

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Definition 1. A polyhedron P in $\mathbb{R}^n$ is a subset defined by a finite number of linear inequalities.

Remark 2. Every polyhedron $P \subseteq \mathbb{R}^n$ can we written as

$$P = \{x \in \mathbb{R}^n : Ax \geqslant b\},$$

for some $m \times n$ matrix A and some $b \in \mathbb{R}^m$.

A linear optimization problem asks to find $\min c \cdot x = \sum_i c_i x_i$ for x belonging to some polyhedron $P \subseteq \mathbb{R}^n$. Some jargon:

- $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is *the cost vector*;
- $c \cdot x$ is *the objective function*;
- every $x \in P$ is called *a feasible solution*;
- if $x^* \in P$ satisfies $c \cdot x^* = \min_{x \in P} c \cdot x$ then x^* is *an optimal solution.*

Definition 3. The standard form problem:

$$\begin{aligned} &\text{minimize} && c \cdot x \\ &\text{subject to} && Ax = b \\ &&& x \geqslant 0. \end{aligned}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$.

Theorem 4. *Every linear problem is* equivalent *to some problem given in the standard form.*

Definition 5. A set $A \subseteq \mathbb{R}^n$ is *convex* if $\lambda x + (1 - \lambda) y \in A$ for every $x, y \in A$ and every $\lambda \in (0, 1)$.

For vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ and scalars $\lambda_j \geqslant 0$ satisfying $\sum_j \lambda_j = 1$, the vector

$$\sum_{j \leqslant k} \lambda_j x^j,$$

is called a convex combination (of those vectors).

By $\mathrm{conv}(x^1, \dots, x^k)$ we denote the *convex hull*, that is the set of all convex combination of those vectors.

Theorem 6. *Every polyhedron is convex. The convex hull* $\mathrm{conv}(x^1, \dots, x^k)$ *is the smallest convex set containing all those vectors.*

Special points and how to compute them

Definition 7. A point x from a convex set $A \subseteq \mathbb{R}^n$ is called an *extreme point* of A if for any distinct $y, z \in A$, if $x = \lambda y + (1-\lambda)z$ then $\lambda = 0$ or $\lambda = 1$.

Definition 8. A point x in a polyhedron $P \subseteq \mathbb{R}^n$ is a *vertex* if there is $c \in \mathbb{R}^n$ such that $c \cdot x < c \cdot y$ for all $y \in P \setminus \{x\}$.

Definition 9. A point x^* in a polyhedron $P \subseteq \mathbb{R}^n$ defined by a system of linear equations and inequalities is a basic solution (BS) if

- x^* satisfies all the equalities;
- there are n linearly independent constraints that are active at x^*.

If, moreover, $x^* \in P$ (i.e. satisfies all the constraints) then it is called a basic feasible solution (BFS).

A constraint $a \cdot x \geqslant b$ (where $a \in \mathbb{R}^n, b \in \mathbb{R}$) is **active at** x^* if $a \cdot x^* = b$.

Theorem 10. *$x \in P$ is an extreme point of P iff x is a vertex of P iff it is BFS.*

Consider a polyhedron P defined in the standard form

$$P = \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\},$$

where A is a matrix $m \times n$ and $b \in \mathbb{R}^m$. We can find all the vertices (=BFS solutions) as follows:

(1) Pick indices $B(1), \ldots, B(m) \leqslant n$ so that the columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent, that is the matrix B consisting of those columns is $m \times m$ and $\det B \neq 0$.
(2) Put $x_i = 0$ for nonbasic indices.
(3) Find $x_B = (x_{B(1)}, \ldots, x_{B(m)})$ solving $Bx_B = b$.
(4) This gives BS; if $x_j \geqslant 0$ for all j then we get BFS.

Such x is degenerate if $x_{B(i)} = 0$ for some i. Otherwise, it is non-degenerate.

Note that if a given simplex has only non-degenerate BFS then there is 1-1 correspondence between bases and those BFS.

In the degenerated case different bases may give the same BFS.

Basic conclusion. Every polyhedron has a finite number of vertices (=BFS).

Theorem 11. *A nonempty polyhedron has at least one vertex iff it contains no lines.*

Theorem 12. *Consider a standard problem:*

$$\begin{array}{ll} \text{minimize} & c \cdot x \\ \text{subject to} & x \in P. \end{array}$$

Suppose that it has an optimal solution and that the polyhedron P has at least one vertex. Then $\min c \cdot x$ *is attained at some vertex of P.*

Changing the vertex

Consider a standard problem:

$$\begin{aligned} &\text{minimize} && c \cdot x \\ &\text{subject to} && Ax = b \\ &&& x \geqslant 0. \end{aligned}$$

We are at some vertex $x \in P$ connected with a basis $B(1), \dots, B(m)$ (of columns of A).

The j-th basic direction: Say that we want to incorporate a non-basic variable j to the basis.

- Find a direction $d \in \mathbb{R}^n$ such that $d_j = 1$ and $d_k = 0$ for other nonbasic variables for which $Ad = 0$.
- This determines the basic part of that direction $d_B = (d_{B(1)}, \dots, d_{B(m)})$:

$$0 = Ad = \sum_{i=1}^{n} d_i A_i = \sum_{i=1}^{m} d_{B(i)} A_{B(i)} + A_j = B d_B + A_j$$

$$d_B = -B^{-1} A_j$$

- We have $A(x + \theta d) = Ax + \theta Ad = b$ so equations hold.

Looking for a new vertex: Suppose that x is non-degenerate, that is all basic coordinates satisfy $x_{B(i)} > 0$.

(1) Then $x + \theta d \geqslant 0$ for small θ.
(2) If $x + \theta d \geqslant 0$ for all $\theta > 0$ then the polyhedron is unbounded — it contains a half-line in that direction.
(3) Otherwise, take the greatest θ for which some i gives $x_{B(i)} + \theta d_{B(i)} = 0$. Remove i from the basis, i comes into it.

Looking for a new vertex; degenerated case: Suppose that x is degenerate, that is $x_{B(i)} = 0$ for some i.

(1) If $d_{B(i)} > 0$ (at each such a case) then we proceed as before.
(2) If $d_{B(i)} \leqslant 0$ at the same time then we are stuck — the direction is not feasible.

Test for optimality

Suppose that we have found a new vertex in the form $y = x + \theta d$. Then

$$c \cdot y - c \cdot x = c \cdot (x + \theta d) - c \cdot c = \theta c \cdot d,$$

so we lowered the objective function if $c \cdot d < 0$.

Recall that

$$d_B = -B^{-1} A_j,$$

so

$$c \cdot d = c_B \cdot d_B + c_j = c_j - c_B \cdot (B^{-1} A_j).$$

Definition.

$$\overline{c_j} = c_j - c_B \cdot (B^{-1} A_j)$$

is called the reduced cost of the jth variable.

Lemma 13. *If $f, g : P \to \mathbb{R}$ are two functions on some sem P and $f - g$ is constant then f and g attain their minima at the same points (if this happens at all).*

Coming back to our standard problem (SP)

$$\begin{aligned} &\text{minimize} && c \cdot x \\ &\text{subject to} && Ax = b \\ & && x \geqslant 0. \end{aligned}$$

denote by $a_1, \ldots, a_m$ the rows of the matrix A

Theorem 14. *If we consider another problem (SP') by changing the cost vector c to*

$$c' = c + \sum_{i=1}^{m} \lambda_i a_i,$$

then (SP) and (SP') have the same solutions.

Recall the formula for reduced costs

$$\overline{c_j} = c_j - c_B \cdot (B^{-1} A_j)$$

$$\overline{c} = c - B^{-1} A$$

Note that if we apply it to the coordinate from the basis then $\overline{c_j} = 0$.

Note also that $\overline{c}$ is a result of adding to c some linear combination of rows of the matrix A.

Theorem 15. *If the vector of reduced costs has nonnegative coordinates then the vertex x (we are at) is optimal.*

Theorem 16. *If the vector of reduced costs $\overline{c}$ satisfies $\overline{c_i} < 0$ for some (necessarily non-basic) coordinate j and the BFS solution x we consider is* **non-degenerate** *then x is not optimal.*