1. **Max flow problem.** Note that a problem of finding a max flow in a graph with a number of sources $s_1, \ldots, s_i$ and many sinks (=targets) $t_1, \ldots, t_j$ can be reduced to the classical case.

   How would you solve the max flow problem in an undirected graph with given capacities of the edges?

2. Matching in bipartite graphs Suppose that we have a graph $G = (V, A)$ where $V = S \cup T$ and every arc in A is of the form (x, y), where $x \in S$ and $y \in T$. A matching in such a graph is an injective function $g : D \to T$ where $D \subseteq S$ and $(x, f(x)) \in A$ for every $x \in D$. Recall that we want to maximize the size of matching (that is $|D|$). It was already suggested how to extend G to the graph $\widetilde{G}$ (by adding the source and the target and so on) so that the FF algorithm can be applied.

   (a) Note that a feasible flow in $\widetilde{G}$ with integer values determines a matching in G.

   (b) Note that an augmenting path in $\widetilde{G}$ defines the so called **alternating chain** in G: it starts at an unmatched vertex in S, ends at an unmatched vertex in T, goes from S to T using a free arc and comes back using an arc from the matching.

   (c) Note that, given some matching, if we find an alternating chain then we can enlarge the matching by one.

   (d) Design a labelling algorithm (working entirely in G) which determines if there are alternating chains.

3. **Theorem (Kőnig).** *In a bipartite graph, the size of a maximal matching equals the minimal number of blocking vertices ($B \subseteq V$ is blocking if every arc either starts in B or ends in it).*

   Prove the theorem by noting that in the extended graph $\widetilde{G}$ (see above) it is just the duality between flows and capacities of cuts.

4. **Hall's marriage theorem.** *In a bipartite graph $G = (V, A)$, where $G = S \cup T$ there is a matching of maximal size $|S|$ if and only if for every $I \subseteq S$ we have $|G[I]| \geqslant |I|$ (here $G[I] = \{y \in T : (x, y) \in A$ for some $x \in I\}$).*

   Prove the theorem by examining the minimal blocking set and using the previous problem.

5. Prove that if every vertex of bipartite graph $G = (S, T, A)$ has the same degree $d > 0$ then there is a matching of the maximal size $|S|$.

6. **The bottleneck problem**. There are n workers and they work along the assembly line (e.g. producing cars). Let $a_{i,j}$ denote the effectiveness of the ith worker at the station

number j. Suppose that we have assigned the ith worker to the station $\sigma(i)$, using some permutation σ. Note that the efficiency of the whole process is then determined by

$$m(\sigma) = \min_{i \leqslant n} a_{i\sigma(i)},$$

as the line must respect the slowest worker. So the problem is to find a permutation σ maximizing $m(\sigma)$.

Consider some permutation σ (e.g. the constant one). To check if it is optimal consider a bipartite graph $G = (T, S, A)$, where $T = S = \{1, 2, \ldots, n\}$ and we draw an arc (i, j) if $a_{ij} > m(\sigma)$.

Prove that σ is not optimal if and only if there is a matching in G of size n. Design an algorithm solving the bottleneck problem.

7. **The caterer problem.** A catering company provides r_i table cloths to a restaurant on each of N consecutive days. New tablecloths can be bougth for the price p per item. The used ones can be send to the express laundry; it makes them unavailable for n days (the cost is l_1 per item) or to the normal laundry; this makes them unaviable for next m days ($m > n$ but the cost l_2 is smaller).

   At the beginning there are no tablecloths available. Find a network flow problem (in the form as in (6) below) that will decide about the optimal plan of delivery. Consider, in particular, the case $N = 5$, $n = 1$, $m = 3$.

   HINT. You will need separate vertices v_i and w_i for clean and dirty tablecloths for the day number i.

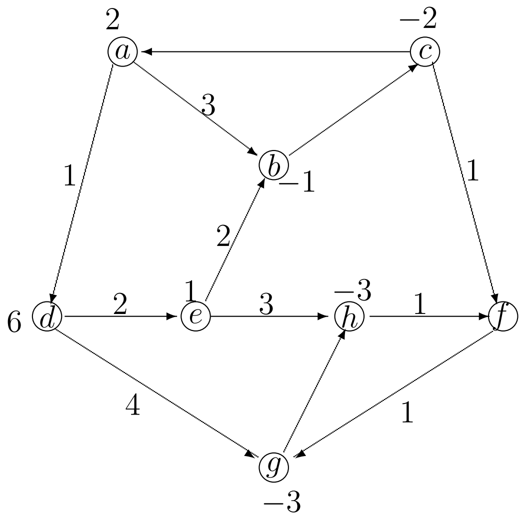
8. **The general network flow problem (GNFP).** Recall that we consider a directed graph $G = (V, A)$ together with a function $V \ni i \to b_i \in \mathbb{R}$ defining an external supply; we assume $\sum_i b_i = 0$. We are also given a function $c : A \to \mathbb{R}_+$, c_{ij} is the cost in the arc $(i, j) \in A$ (no limit on capacities of the arc).

   Recall that a flow $f = (f(i,j))_{(i,j)\in A}$ is **feasible** if $0 \leqslant f(i, j)$ for every arc $(i, j) \in A$ and $\sum_{j\in \mathrm{In}(i)} f(j, i) + b_i = \sum_{j\in \mathrm{Out}(i)} f(i, j)$ for every vertex i. We want to minimize $\sum_{(i,j)\in A} c_{ij} f(i, j)$ over all feasible flows.

   Check that this is a linear problem in its standard form when we consider the incidence matrix $M = (m_{ij})$; its rows correspond to vertices and columns represent arcs. We put $m_{ij} = 1$ if the jth arc starts at i and $m_{ij} = -1$ if the jth arc ends at i; other entries are zero. Prove that $Mf = b$.

9. Using the form of GNFP mentioned above (and general duality for linear problems) find the dual problem of GNFP.

10. **Exercises on GNFP.** Consider the network (with b_i and c_{ij} given)



Find some spanning tree and the basic flow associated to it (feasible or not). Once you find some feasible basic flow, reduce the costs to see if it is optimal.