# Flows in networks and duality

We now think that $V = \{1, 2, \dots, n\}$.

**The network.** Given a directed graph $G = (V, A)$ together with

- a function $V \ni i \to b_i \in \mathbb{R}$ defining an external supply; we assume $\sum_i b_i = 0$;
- a function $c : A \to \mathbb{R}_+$, $c_{ij}$ is the cost in the arc $(i, j) \in A$

**Definition.** A flow $f = (f(i,j))_{(i,j)\in A}$ is **feasible** if

(i) $0 \leqslant f(i,j)$ for every arc $(i,j) \in A$;

(ii) $\sum_{j\in \mathrm{In}(i)} f(j,i) + b_i = \sum_{j\in \mathrm{Out}(i)} f(i,j)$ for every vertex $i$.

**GNFP.** Minimize

$$\sum_{(i,j)\in A} c_{ij} f(i,j),$$

over all feasible flows.

# Dual of GNFP

Consider a problem (D)

$$\max \sum_{i \leqslant n} b_i y_i \quad \text{subject to}$$

$$y_i - y_j \leqslant c_{ij} \text{ for all } (i,j) \in A.$$

**Theorem (weak duality).** If $y$ is a feasible solution of (D) then

$$b \cdot y = \sum_{i \in V} b_i y_i \leqslant \sum_{(i,j) \in A} c_{ij} f(i,j),$$

for every feasible flow $f$.

**Complementarity.** If we find feasible $f$ and $y$ such that

$$c_{ij} f(i,j) = (y_i - y_j) f(i,j) \text{ for every } (i,j) \in A,$$

then $f$ and $y$ are optimal.

## Approach to the dual problem

$$\max \sum_{i \leqslant n} b_i y_i \quad \text{subject to}$$

$$\underline{y_i - y_j \leqslant c_{ij}} \text{ for all } (i,j) \in A.$$

$y = (y_1, y_2, \dots, y_n)$
$y + a = (y_1 + a, y_2 + a, \dots)$

**Observe that:** $y = 0$ is a feasible solution and for every feasible $y$, $y + a$ is feasible too.

Fix feasible solution $y$

**Jargon.** An arc $(i,j)$ is **saturated** (with respect to $y$) if $\underline{c_{ij} = y_i - y_j}$.
A set $S \subseteq V$ is **balanced** if there is no saturated arc leaving $S$.

**Lemma.** If $y$ is feasible, $S$ is balanced then $\underline{y^* = y + \theta\chi_S}$ is also feasible, where

$$\theta = \min\{(c_{ij} - (y_i - y_j)) : (i,j) \in A, i \in S, j \notin S\} > 0.$$

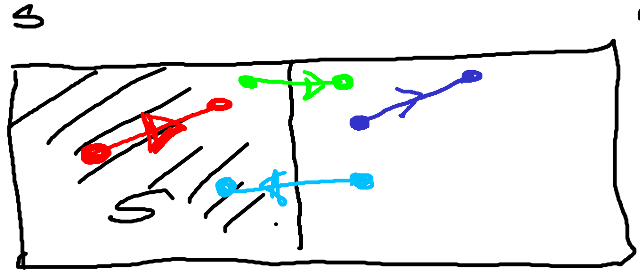
$\chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

$\underline{y^* = y + \theta\chi_S}$

$y^*_i = \begin{cases} y_i + \theta & \text{if } i \in S \\ y_i & \text{if } i \notin S \end{cases}$

$V = \{1, 2, \dots, n\}$

$y_i^* - y_j^* =$
$= (y_i + \theta) - (y_j + \theta) =$
$= y_i - y_j \leq c_{ij}$



$\underset{\underset{y_i - y_j}{\|}}{y_i^* - y_j^*} \leq c_{ij}$

$y_i^* - y_j^* = y_i - (y_j + \theta) = y_i - y_j - \theta \leq y_i - y_j \leq c_{ij}$

$y^* = y + \theta x_S$

## Outline of the algorithm for the dual problem

**When $y^*$ is better:** If and only if

$$\sum_{i\in V} b_i y_i^* - \sum_{i\in V} b_i y_i = \boxed{\theta \sum_{i\in S} b_i} > 0.$$

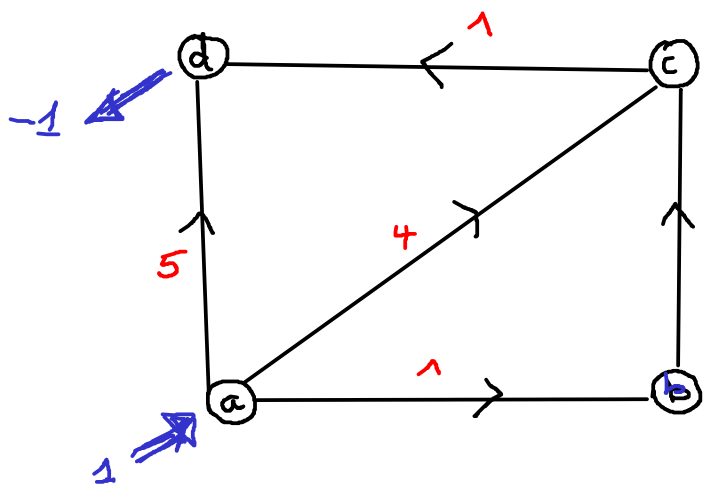
**Idea**

(1) Start from some feasible $y$.

(2) Look for balanced $S \subseteq V$ such that $\sum_{i\in S} b_i > 0$. If there are no such $S$ then STOP — $y$ is optimal.

(3) Change $y$ to $y^*$ and repeat.

# Example




$c_{ij}$ in red

saturated = $c_{ij} = y_i - y_j$

$y = (0,0,0,0)$ $S = \{a\}$ $\Theta = 1$

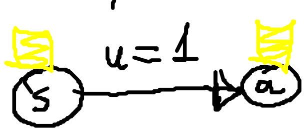
$y' = (1,0,0,0)$ $S = \{a,b\}$ $\Theta = 1$

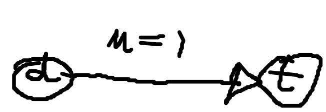
$y'' = (2,1,0,0)$ $S = \{a,b,c\}$ $\Theta = 1$

$y''' = (3,2,1,0)$ optimal.

$$\Theta = \min \left( c_{ij} - (y_i - y_j) \right)$$

Ilustracje algorytmu



# Primar-dual algorithm

GNFP

$$\min \sum_{(i,j)\in A} c_{ij} f(i,j)$$

D

$$\max \sum_i b_i y_i$$

**Outline**

(1) Start from some $y$ feasible for the dual problem.

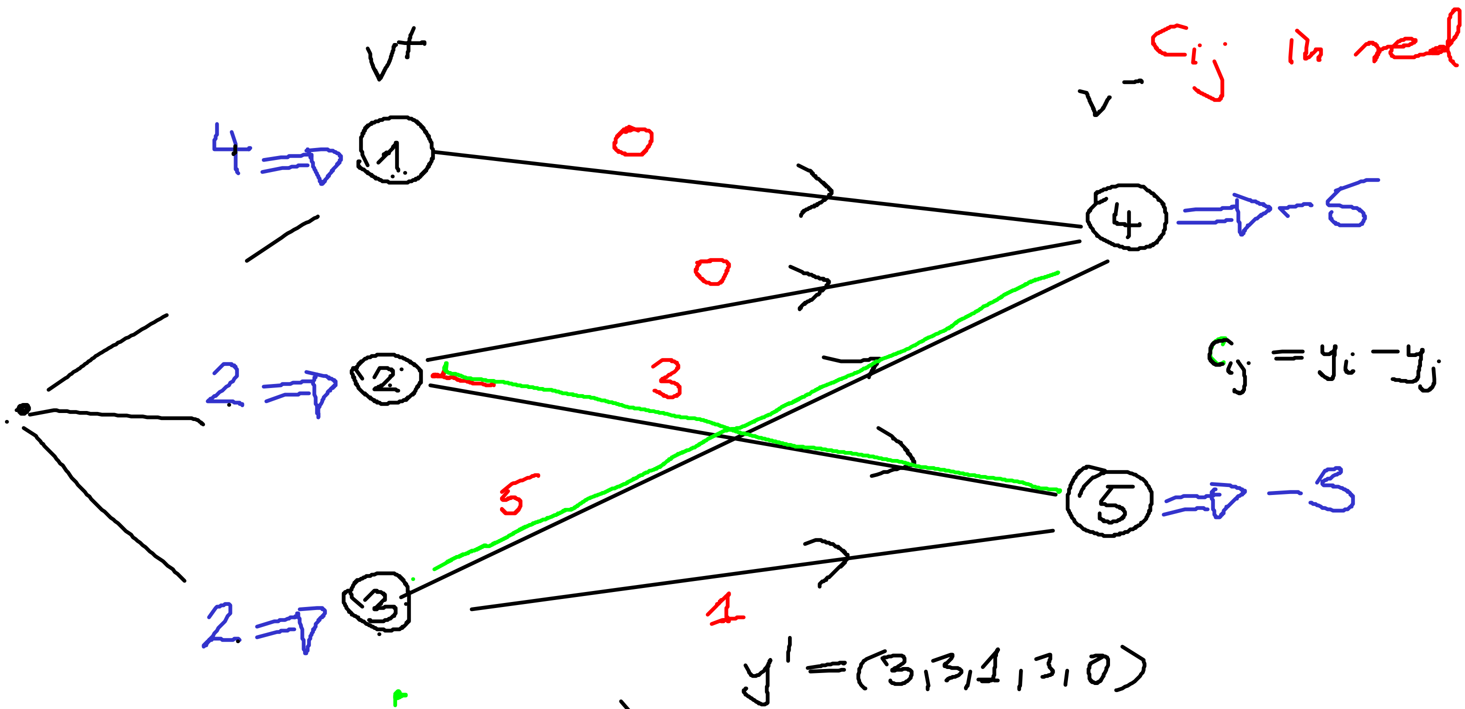
(2) Use FF to check how much can flow from the source vertices $V^+$ to target vertices $V^-$ through the graph consisting only of saturated arcs.

(3) If the volume of the flow is equal to $\underbrace{\Sigma_{i\in V^+} b_i}$ then STOP (we have an optomal flow).

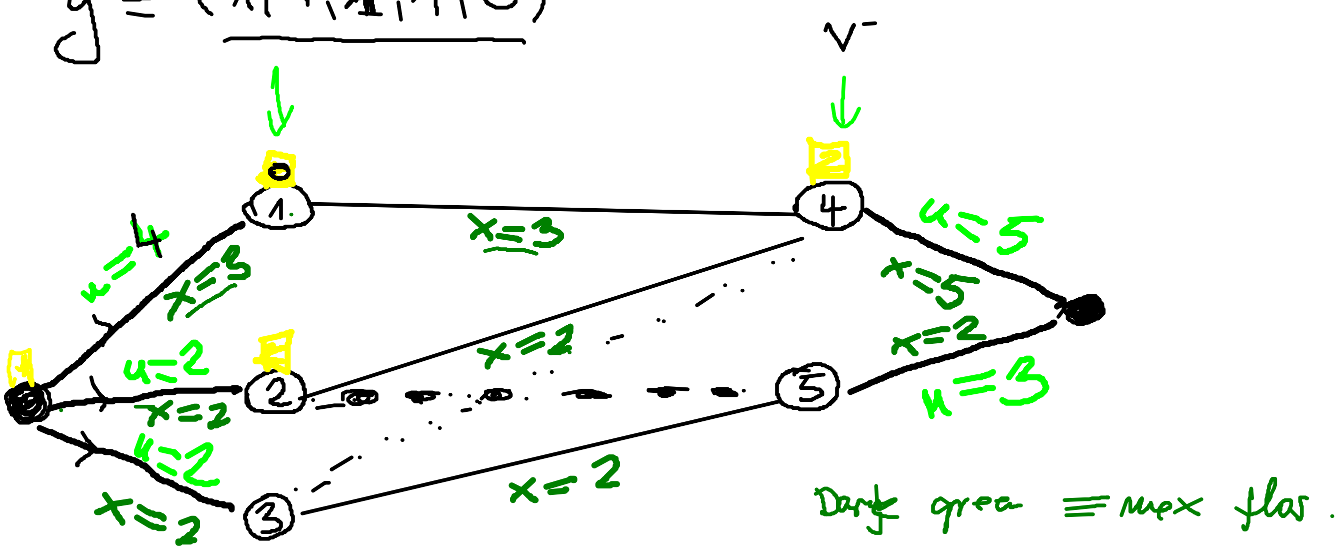
(4) Otherwise LA finds a balanced $S$ with $\boxed{\underbrace{\Sigma_{i\in S} b_i} > 0}$ while checking that the flow is maximal.

(5) Change $y$ to $\underline{y^* = y + \theta}\chi_S$; GoTo (2).

# Example




$y = (1,1,1,1,0)$

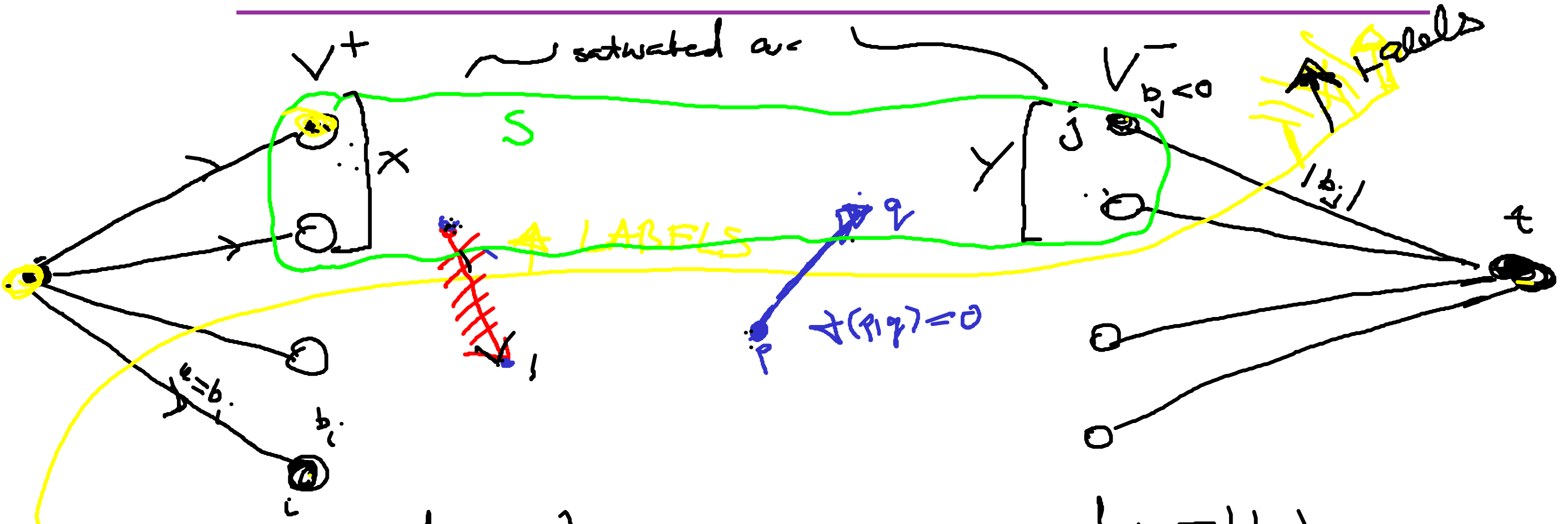



$\boxed{S = \{1,2,4\}}$ $\quad \theta = 2$

$y = (1,1,1,0,0)$

$$y' = y + 2\,\underbrace{(1,1,0,1,0)}_{\delta} = \underline{(3,3,1,3,0)}$$

## What happens here?




$f_{si} = b_i$ $\qquad$ $f_{jt} = |b_j|$

We want to prove: $\boxed{\sum_{i \in S} b_i > 0}$

and the set S (in green) is balanced.

X przepływ przez wierzchołki $V^+ \cap S$

Y przepływ przez wierzchoł. $V^- \cap S$

Wtedy $X = Y$

$X \leq \sum_{i \in V^+ \cap S} b_i$ $\qquad$ $Y = \sum_{i \in V^- \cap S} |b_i|$

stąd $\sum_{i \in S} b_i > 0$.