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Convergence of Semigroups of Complex Measures on a Lie Group

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Abstract A theorem of Siebert in its essential part asserts that if $\mu_n(t)$ are semigroups of probability measures on a Lie group G, and P_n are the corresponding generating functionals, then

$$\big\langle \mu_n(t), f\big\rangle \underset{n}{\to} \big\langle \mu_0(t), f\big\rangle, \quad f \in C_b(G),\ t > 0,$$

implies

$$\langle \pi_{P_n} u, v\rangle \underset{n}{\to} \langle \pi_{P_0} u, v\rangle, \quad u \in C^\infty(E,\pi),\ v \in E,$$

for every unitary representation π of G on a Hilbert space E, where $C^\infty(E,\pi)$ denotes the space of smooth vectors for π.

The aim of this note is to give a simple proof of the theorem and propose some improvements, the most important being the extension of the theorem to semigroups of complex measures. In particular, we completely avoid employing unitary representations by showing simply that under the same hypothesis

$$\langle P_n, f\rangle \underset{n}{\to} \langle P_0, f\rangle,$$

for bounded twice differentiable functions f.

As a corollary, the above thesis of Siebert is extended to bounded strongly continuous representations of G on Banach spaces.

Keywords Semigroups of measures · Dissipative distributions · Hunt theory · Lie groups · Unitary representations

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1 Introduction

Let E be a Banach space and

$$\boldsymbol{A}_n : \mathcal{D}_n \to E, \quad \mathcal{D}_n = \operatorname{dom}(\boldsymbol{A}_n) \subset E,$$

infinitesimal generators of strongly continuous contraction semigroups $e^{t\boldsymbol{A}_n}$ on E. It is classical that

$$e^{t\boldsymbol{A}_n} f \underset{n}{\to} e^{t\boldsymbol{A}_0} f, \quad f \in E,\ t > 0, \tag{1.1}$$

is equivalent to

$$(\lambda I - \boldsymbol{A}_n)^{-1} f \underset{n}{\to} (\lambda I - \boldsymbol{A}_0)^{-1} f, \quad f \in E,\ \operatorname{Re}\lambda > 0. \tag{1.2}$$

(See, e.g. Yosida [15], IX.12.) Furthermore, if there exists a common core domain $\mathcal{D} \subset \mathcal{D}_n$, then

$$\boldsymbol{A}_n f \underset{n}{\to} \boldsymbol{A}_0 f, \quad f \in \mathcal{D}, \tag{1.3}$$

implies (1.1). See Kato [10], Theorem VIII.1.5. It may happen, however, that (1.1) holds with $\mathcal{D}_n = \mathcal{D}_1$, for $n \geq 1$, while $\mathcal{D}_0 \cap \mathcal{D}_1 = \emptyset$ (see Engel–Nagel [3], Chap. III, Counterexample 5.10) so in general (1.1) does not imply (1.3).

A remarkable property of convolution semigroups of measures is that (1.1) in a way does imply (1.3). Namely, if $\mu_n(t)$ is a sequence of semigroups for probability measures on a Lie group G, and P_n are the corresponding generating functionals, then

$$\big\langle \mu_n(t), f \big\rangle \underset{n}{\to} \big\langle \mu_0(t), f \big\rangle, \quad f \in C_b(G),\ t > 0, \tag{1.4}$$

implies

$$\langle \pi_{P_n} u, v \rangle \underset{n}{\to} \langle \pi_{P_0} u, v \rangle, \quad u \in C^\infty(E, \pi),\ v \in E,$$

for every unitary representation π of G on a Hilbert space E, where $C^\infty(E, \pi)$ denotes the space of smooth vectors for π, see Siebert [13], Proposition 6.4.

E. Siebert's theorem on convergence of continuous convolution semigroups and generating functionals (on Lie-projective groups) was proved in the context of commuting triangular arrays and convergence criteria. The main result was implicit in a sequence of results. Later on, starting with Yu. Khokhlov [11], the convergence theorem has been noticed, appreciated and given applications. Subsequently G. Pap [12] gave a new proof for the Lie group case still relying on Siebert's Propositions 6.3 and 6.4.

As mentioned above, Siebert formulated his theorem on Lie-projective groups, where the Lie group case is the crucial step to make. By structural properties the result extends to locally compact groups. This generalisation is due to Hazod [5]:

Theorem 1.5 *Let $\mu_n(t)$ be a sequence of continuous semigroups of probability measures on a locally compact group G. Let P_n be the corresponding sequence of gener-*

ating functionals. Then

$$\langle \mu_n(t), f\rangle \to \langle \mu_0(t), f\rangle, \quad f \in C_b(G),$$

if and only if

$$\langle P_n, f\rangle \to \langle P_0, f\rangle, \quad f \in \mathcal{E}(G),$$

where $\mathcal{E}(G)$ is the Bruhat class of regular functions.

The *if* implication follows from the general theory of contraction semigroups mentioned above (implication of (1.1) by (1.3)). The aim of this note is to give a simple proof of the *only if* implication in the setting of a Lie group and propose some improvements. The striking simplicity of our proof as compared to that of Siebert (see Siebert [13] and also a sketch of the proof in Hazod–Siebert [6], Theorem 2.0.12) is our main argument for the presentation. The main idea is that the norm of a generating functional on the Hunt space $\mathcal{C}^2(G)$ can be controlled by its action on coordinate functions. This helps to eliminate any reference to unitary representations so prominent in Siebert [13]. Our method works also in the case of continuous convolution semigroups of complex measures.

As an introduction to the theory of semigroups of measures on Lie groups we recommend Duflo [2], Faraut [4], Hulanicki [8], and Hunt [9]. The reader may also wish to consult Hazod–Siebert [6] or Heyer [7].

I feel greatly indebted to the referee whose apt critique and extensive comments have been very helpful and substantially contributed to the improvement of the presentation.

2 Preliminaries

Let G be a Lie group with a rightinvariant Haar measure dx. Let $C_c^\infty(G)$ denote the space of smooth functions on G with compact support. To fix notation let us list briefly the most basic formulae. For $f \in C_c^\infty(G)$, let

$$f^\#(x) = f\left(x^{-1}\right)\Delta\left(x^{-1}\right), \quad \widetilde{f}(x) = f\left(x^{-1}\right),$$

where Δ is the modular function on G. Then

$$f \star g(x) = \int f\left(xy^{-1}\right)g(y)\,dy,$$

and

$$\widetilde{f} \star g(x) = \int f(y)g(yx)\,dy, \qquad f \star g^\#(x) = \int f(xy)g(y)\,dy.$$

Let

$$\langle f, g\rangle = \int f(x)g(x)\,dx.$$

We have

$$\langle f \star g, h \rangle = \langle f, h \star g^{\#} \rangle = \langle g, \widetilde{f} \star h \rangle.$$

If P is a distribution on G, then

$$\widetilde{P} \star f(x) = \langle P, f_x \rangle, \quad f \in C_c^\infty(G),$$

where $f_x(y) = f(yx)$. If X is a leftinvariant vector field on G, then

$$Xf(x) = f \star \widetilde{X}(e)(x),$$

where $X(e)$ is a distribution supported at the identity e acting by

$$\big\langle X(e), f \big\rangle = Xf(e), \quad f \in C_c^\infty(G).$$

Let $\{X_j\}_{j=1}^d$ be a basis of leftinvariant vector fields on G. Let $C_b(G)$ denote the space of bounded continuous functions on G. It is a Banach space with the supremum norm $\|\cdot\|_\infty$. Its closed subspace $\mathcal{C}(G)$ of functions with a limit at infinity will play a major role here. Let us distinguish two other subspaces

$$C_b^2(G) = \big\{ f \in C_b(G) : \|f\|_{C^2} < \infty \big\},$$

and

$$\mathcal{C}^2(G) = \big\{ f \in \mathcal{C}(G) : \|f\|_{C^2} < \infty \big\},$$

where

$$\|f\|_{C^2} = \max_{|\alpha| \le 2} \big\| X^\alpha f \big\|_\infty.$$

3 Generating Functionals

Let us recall that a one-parameter family $\mu(t), t > 0$, of complex Borel measures on G is called *a continuous semigroup of measures* if

(a) $\mu(t) \star \mu(s) = \mu(t+s),\ t, s > 0$,
(b) $\langle \mu(t), f \rangle \to f(e)$, for $f \in C_b(G)$, if $t \to 0$,
(c) $\|\mu(t)\| \le 1,\ t > 0$.

If $\mu(t)$ is a continuous semigroup of measures, then the limit

$$\langle P, f \rangle = \lim_{t \to 0} \frac{\langle \mu(t), f \rangle - f(e)}{t}$$

exists for every $f \in C_c^\infty(G)$ and defines a distribution P called the generating functional. A generating functional of a continuous semigroup of measures has the property

$$\operatorname{Re}\langle P, f \rangle \le 0 \tag{3.1}$$

for $f \in C_c^\infty(G)$ such that $f(e) = \|f\|_\infty$. Such a distribution is called *dissipative*.

On the other hand, for every dissipative distribution P on G, there exists a unique continuous semigroup of measures $\mu(t)$ for which P is the generating functional.

Let us add that the measures in the semigroup are subprobability measures if and only if the generating functional P is *a generalised Laplacian*, that is, P is real and

$$\langle P, f \rangle \leq 0 \tag{3.2}$$

for every real $f \in C_c^\infty(G)$ such that $f(e) = \sup_{x \in G} f(x)$.

Let us denote by $\mathcal{P}(G)$ the cone of dissipative distributions and by $\mathcal{P}_0(G)$ the subcone of generalised Laplacians. An immediate consequence of the definition is that every $P \in \mathcal{P}(G)$ coincides with a Radon measure η on the open set $G \setminus \{e\}$. If $P \in \mathcal{P}_0(G)$ the measure is nonnegative. In general η is unbounded. It is bounded, however, outside any neighbourhood of the origin. More specifically, if U is a neighbourhood of e, $\phi \in C_c^\infty(U)$ with $0 \leq \phi \leq 1$ and $\phi(e) = 1$, then for $f \in C_c^\infty(G)$ supported in $G \setminus \bar{U}$,

$$\left| \int f(x) \eta(dx) \right| \leq \left| \langle P, \phi - 1 \rangle \right| \|f\|_\infty \tag{3.3}$$

(see Faraut [4], Proposition II.2 and below). Even though Faraut [4] is concerned only with the $\boldsymbol{R}^n$ the argument is fully applicable in the case of a Lie group. This is because dissipativity does not depend on the group structure. Moreover, the decomposition of a dissipative distribution as a sum of a compactly supported distribution with arbitrarily small support and a bounded measure (the argument here is no different from that on $\boldsymbol{R}^n$) makes the whole matter in fact local.

Lemma 3.4 (Faraut [4], Proposition IV.1) *Any $P \in \mathcal{P}(G)$ extends to a continuous linear functional on $\mathcal{C}^2(G)$, and the extension preserves property* (3.1) (*resp. property* (3.2)) *on the larger class of functions.*

Since P is a bounded measure away from the identity, it can be also regarded as a linear form on the space $C_b^2(G)$. For the sake of simplicity of notation we shall write $\eta(dx) = P(dx)$.

Remark 3.5 If $f \in \mathcal{C}^2(G)$ and

$$f(e) = X_j f(e) = X_j X_k f(e) = 0, \quad 1 \leq j, k \leq d,$$

then

$$\langle P, f \rangle = \int_{G \setminus \{e\}} f(x) P(dx), \tag{3.6}$$

for $P \in \mathcal{P}(G)$. In fact, (3.6) holds for f vanishing in a neighbourhood of e and extends by continuity of P on $\mathcal{C}^2(G)$ to f as above.

The following lemma offers an extension of (3.3).

Lemma 3.7 *Let $P = P_1 + iP_2 \in \mathcal{P}(G)$, where P_1, P_2 are real. Then, for every* [0, 1]-*valued $f \in \mathcal{C}^2(G)$ with $f(e) = 0$,*

$$\int f(x)|P_1|(dx) \le \langle P_1, f - 1\rangle, \tag{3.8}$$

and

$$\int f(x)|P_2|(dx) \le \langle P_1, f - 2\rangle. \tag{3.9}$$

Consequently,

$$\int f(x)|P|(dx) \le 3\big(\big|\langle P, f\rangle\big| + \big|\langle P, 1\rangle\big|\big).$$

Proof Let $f \in \mathcal{C}^2(G)$ be a [0, 1]-valued function with $f(e) = 0$. Then,

$$\int f(x)|P_j|(dx) = \sup_g \langle P_j, gf\rangle, \quad j = 1, 2, \tag{3.10}$$

where the supremum is taken over real $g \in C_c^\infty(G \setminus \{e\})$ with $|g| \le 1$. However, if $h = 1 - f + gf$, then $h(e) = 1 = \|h\|_\infty$ so that, by (3.1),

$$\langle P_1, h\rangle = \mathrm{Re}\langle P, h\rangle \le 0,$$

and consequently

$$\langle P_1, gf\rangle \le \langle P_1, f - 1\rangle,$$

which, by (3.10), implies (3.8).

Now, let $k = 1 - (gf)^2 - igf$. Then, $k(e) = 1 = \|k\|_\infty$ so that, by (3.1), $\mathrm{Re}\langle P, k\rangle \le 0$, which, by (3.8), implies

$$\begin{aligned}\langle P_2, gf\rangle &\le \big\langle P_1, (gf)^2 - 1\big\rangle = \int (gf)(x)^2 P_1(dx) + \langle P_1, -1\rangle \\ &\le \int f(x)|P_1|(x) + \langle P_1, -1\rangle \le \langle P_1, f - 2\rangle.\end{aligned}$$

Again, by (3.10), (3.9) follows. Finally,

$$\int f(x)|P|(dx) \le \langle P_1, 2f - 3\rangle \le 3\big(\big|\langle P, f\rangle\big| + \big|\langle P, 1\rangle\big|\big).$$

□

Remark 3.11 If P is a generalised Laplacian, then, for every f as above,

$$\int f(x)P(dx) \le \langle P, f\rangle.$$

In fact, for $g \in C_c^\infty(G \setminus \{e\})$, $0 \le g \le 1$, we have

$$\int gf(x)P(dx) = \langle P, gf\rangle = \langle P, f\rangle + \big\langle P, (g - 1)f\big\rangle \le \langle P, f\rangle$$

since $(g-1)f$ attains its maximal value at e and so (3.2) applies. Consequently, the desired estimate follows.

It is convenient to introduce coordinate functions near the identity. Let $\Phi_k \in C_c^\infty(G)$ be real functions such that

$$X_j \Phi_k(e) = \delta_{jk}, \quad 1 \le k, j \le d.$$

There exists a $[0,1]$-valued function $\Phi^2 \in C^\infty(G)$ such that

(a) $\Phi^2 = \sum_{k=1}^d \Phi_k^2$ in a neighbourhood of e,
(b) $\Phi^2(x) = 1$ outside a compact neighbourhood of e,
(c) $\Phi^2(x) > 0$, for $x \neq e$.

The function Φ^2 is called *a Hunt function* (see Hazod-Siebert [6], p. 187 and Heyer [7], Lemma 4.1.9). We have the following Taylor estimate (cf., e.g. Hulanicki [8], (1.1)):

$$\left| f(x) - f(e) - \sum_{j=1}^d X_j f(e)\Phi_j(x) \right| \le C\|f\|_{C^2}\Phi^2(x) \tag{3.12}$$

for $f \in \mathcal{C}^2(G)$. The constant C here and throughout the paper is a generic constant which may vary from statement to statement.

Remark 3.13 Let $\Phi_{jk} = \Phi_j\Phi_k$.There exists a constant $C > 0$ such that

$$\left|\langle P, \Phi_{jk}\rangle\right| \le C\langle P_1, \Phi^2 - 1\rangle, \quad 1 \le j, k \le d.$$

This is proved in the same way as Lemma 3.7 by considering the functions

$$h = 1 - \Phi^2 \pm c\Phi_{jk}, \qquad k = 1 - \Phi^2 \pm ic\Phi_{jk}$$

for $c > 0$ small enough.

The following proposition is well-known at least in the case of generalised Laplacians (see, e.g. Siebert [14], 2.5). We include a proof because the result is vital for our main theorem.

Proposition 3.14 *There exists a constant $C > 0$ such that, for every dissipative distribution P,*

$$\|P\| \le C\left(\left|\langle P, \Phi^2\rangle\right| + \sum_{j=1}^d \left|\langle P, \Phi_j\rangle\right| + \left|\langle P, 1\rangle\right| \right),$$

where $\|P\|$ is the norm of P as a linear functional on $\mathcal{C}^2(G)$.

Proof For $f \in \mathcal{C}^2(G)$, let

$$f_1(x) = f(e) + \sum_{j=1}^d X_j f(e)\Phi_j(x), \qquad f_2(x) = f_1(x) + \frac{1}{2}\sum_{j,k=1}^d X_j X_k f(e)\Phi_{jk}.$$

Then,

$$\big|\langle P, f_1\rangle\big| \le \left(\sum_{j=1}^{d}\big|\langle P, \Phi_j\rangle\big| + \big|\langle P, 1\rangle\big|\right)\cdot \|f\|_{C^2}$$

and, by (3.6), (3.12), and Lemma 3.7,

$$\begin{aligned}\big|\langle P, f-f_2\rangle\big| &= \left|\int (f-f_2)(x)P(dx)\right| \\ &\le \int |f-f_1|(x)|P|(dx) + \frac{1}{2}\sum_{j,k}\int \big|\Phi_{jk}(x)\big||P|(dx)\cdot\|f\|_{C^2} \\ &\le C\int \Phi^2(x)|P|(dx)\cdot\|f\|_{C^2} \le C\big(\big|\langle P, \Phi^2\rangle\big| + \big|\langle P, 1\rangle\big|\big)\cdot\|f\|_{C^2}.\end{aligned}$$

Finally,

$$\langle P, f\rangle = \langle P, f-f_2\rangle + \langle P, f_1\rangle + \frac{1}{2}\sum_{j,k} X_j X_k f(e)\langle P, \Phi_{jk}\rangle,$$

where, by Remark 3.13,

$$\big|\langle P, \Phi_{jk}\rangle\big| \le C\big(\big|\langle P, \Phi^2\rangle\big| + \big|\langle P, 1\rangle\big|\big)$$

so that our assertion follows. □

Remark 3.15 The estimates

$$\int \big|\Phi_{jk}(x)\big|\,|P|(dx) \le C\int \Phi^2(x)|P|(dx)$$

and

$$\big|\langle P, \Phi_{jk}\rangle\big| \le C\big(\big|\langle P, \Phi^2\rangle\big| + \big|\langle P, 1\rangle\big|\big)$$

look very much alike, but in fact are of a different nature. The first is trivial whereas the latter requires dissipativity.

Corollary 3.16 *Let $P_n \in \mathcal{P}(G)$. If $\langle P_n, f\rangle \to 0$ for every $f \in \mathcal{C}^2(G)$, then $\|P_n\| \to 0$.*

4 Convergence

Let $\mu(t)$ be a continuous semigroup of measures on G with the generating functional P. We are going to regard such semigroups as acting by convolutions on the left on $\mathcal{C}(G)$. The fundamental theorem we are going to take advantage of is the following.

Theorem 4.1 *The convolution operators*

$$T(t)f(x) = \widetilde{\mu}(t) \star f(x) = \int f(yx)\mu(t)(dy) \tag{4.2}$$

form a strongly continuous contraction semigroup on the Banach space $\mathcal{C}(G)$. The infinitesimal generator of $T(t)$ is the convolution operator

$$\boldsymbol{P} f(x) = \widetilde{P} \star f(x) = \langle P, f_x \rangle$$

for which $C_c^\infty(G)$ is a core domain.

For a proof we refer to Duflo [2] where the theorem is proved in a more general setting of the Banach space representations of G (see Proposition 5.2 below and a more precise reference given there). A more direct proof can be obtained by the line of argument as in Hunt [9] or Hulanicki [8] where, however, only the case of probability measures is dealt with.

Remark 4.3 Recall that if $T_n(t)$ are strongly continuous contraction semigroups on a Banach space E and

$$T_n(t)u \underset{n}{\to} T_0(t)u, \quad u \in E,\ t > 0,$$

then for every fixed $u \in E$ the convergence is uniform in $0 \le t \le 1$. (See, e.g. Yosida [15], Theorem IX.12.1.)

Theorem 4.4 *Let $P_n \in \mathcal{P}(G)$, $n = 0, 1, 2, \ldots$. Denote by $\mu_n(t)$ the semigroup of measures generated by P_n. If*

$$\left\langle \mu_n(t), f \right\rangle \underset{n}{\to} \left\langle \mu_0(t), f \right\rangle, \quad f \in C_b(G),\ t > 0,$$

then, for every $f \in \mathcal{C}^2(G)$,

$$\langle P_n, f \rangle \underset{n}{\to} \langle P_0, f \rangle.$$

Proof Let

$$T_n f(x) = \widetilde{\mu}_n(t) \star f, \quad f \in \mathcal{C}(G).$$

Then, for every n, the operators $T_n(t)$ form a strongly continuous contraction semigroup on $\mathcal{C}(G)$. Furthermore, for each $t > 0$, the family $\mu_n(t)$ is uniformly tight, which follows by the Prochorov theorem for complex measures (Bogachev [1], Theorem 8.6.2). Consequently,

$$\lim_{n\to\infty} \left\| T_n(t)f - T_0(t)f \right\|_\infty = 0, \quad f \in \mathcal{C}(G),\ t > 0.$$

The operators $T_n(t)$ commute with leftinvariant derivatives X^α. Therefore, the Banach space $\mathcal{C}^2(G)$ is invariant under the semigroup $T_n(t)$ which is a strongly continuous contraction semigroup on this new Banach space. Similarly,

$$\lim_{n\to\infty} \left\| T_n(t)f - T_0(t)f \right\|_{\mathcal{C}^2} = 0, \quad f \in \mathcal{C}^2(G),\ t > 0.$$

Let $f \in C_c^\infty(G)$. We have

$$
\begin{aligned}
\langle \mu_0(t) - \mu_n(t), f \rangle &= \langle \mu_0(t) - \delta_0, f \rangle - \langle \mu_n(t) - \delta_0, f \rangle \\
&= \int_0^t \langle \mu_0(s) \star P_0, f \rangle ds - \int_0^t \langle \mu_n(s) \star P, f \rangle ds \\
&= \int_0^t \langle P_0, \widetilde{\mu}_0(s) \star f \rangle ds - \int_0^t \langle P_n, \widetilde{\mu}_n(s) \star f \rangle ds \\
&= \int_0^t \langle P_0, T_0(s) f - f \rangle ds - \int_0^t \langle P_n, T_n(s) f - f \rangle ds \\
&\quad + t \langle P_0 - P_n, f \rangle \\
&= \int_0^t \langle P_0 - P_n, T_0(s) f - f \rangle ds \\
&\quad + \int_0^t \langle P_n, T_0(s) f - T_n(s) f \rangle ds + t \langle P_0 - P_n, f \rangle,
\end{aligned}
$$

whence

$$
\begin{aligned}
\langle P_0 - P_n, f \rangle = \frac{1}{t} \langle \mu_0(t) - \mu_n(t), f \rangle + \frac{1}{t} \int_0^t \langle P_n - P_0, T_0(s) f - f \rangle ds \\
+ \frac{1}{t} \int_0^t \langle P_n, T_n(s) f - T_0(s) f \rangle ds. \qquad (4.5)
\end{aligned}
$$

Assume for the moment that

$$
\sup_n \| P_n \| \leq M < \infty. \qquad (4.6)
$$

By the remarks at the beginning of the proof, for every $f \in C_c^\infty(G)$ and every $t > 0$,

$$
\lim_{n \to \infty} \| T_n(t) f - T_0(t) f \|_{C^2} \to 0, \qquad \lim_{s \to 0} \| T_0(s) f - f \|_{C^2} = 0. \qquad (4.7)
$$

Since, by Remark 4.3, the convergence of the semigroups on $\mathcal{C}^2(G)$ is uniform for $0 \leq s \leq t \leq 1$, (4.6) implies the desired convergence of generating functionals. In fact, by (4.5),

$$
\begin{aligned}
\left| \langle P_0 - P_n, f \rangle \right| \leq \frac{1}{t} \left| \langle \mu_0(t) - \mu_n(t), f \rangle \right| \\
+ \frac{2M}{t} \int_0^t \| T_0(s) f - f \|_{C^2} ds \\
+ \frac{M}{t} \int_0^t \| T_n(s) f - T_0(s) f \|_{C^2} ds.
\end{aligned}
$$

Now we pick $t > 0$ small enough to make the middle term small, then fix t and take n large enough to make the remaining terms small, which can be done by (4.7). Thus, it remains to show that (4.6) holds under the hypothesis of the theorem.

In fact, assume *a contrario* that this is not true. Then there exists a sequence of integers n_k such that $\alpha_k = \|P_{n_k}\| \to \infty$. The generating functionals $Q_k = \alpha_k^{-1} P_{n_k}$ satisfy

$$\|Q_k\| = 1, \quad k \in \boldsymbol{N}. \tag{4.8}$$

However, by dividing both sides of (4.5) (with $n = n_k$) by α_k and arguing as above, we see that

$$\langle Q_k, f\rangle \to 0, \quad f \in C_c^\infty(G),$$

which, by Corollary 3.16, implies that $\|Q_k\| \to 0$. This contradicts (4.8) and completes the proof for $f \in C_c^\infty(G)$.

However, $C_c^\infty(G)$ is dense in $\mathcal{C}^2(G)$ and, by 4.6, the functionals P_n are uniformly continuous on $\mathcal{C}^2(G)$ so the convergence must hold for all $f \in \mathcal{C}^2(G)$. □

Corollary 4.9 *Let P_n satisfy the hypothesis of Theorem 4.4. Then, for every $\varepsilon > 0$, there exists a relatively compact neighbourhood U of e such that, for $n \in \boldsymbol{N}$,*

$$\int_{G\setminus U} |P_n|(dx) < \varepsilon.$$

Consequently,

$$\lim_{n\to\infty} \int_G f(x) P_n(dx) = \int_G f(x) P_0(dx)$$

for every bounded continuous function f vanishing in a neighbourhood of e.

Proof Let V be a relatively compact neighbourhood of e such that $\int_{G\setminus V} |P_0|(dx) < \varepsilon$. Take a $[0, 1]$-valued $\phi \in C_c^\infty(G)$ and a relatively compact neighbourhood U of e such that $\phi = 1$ on V and $\operatorname{supp}\phi \subset U$. Then, by (3.3) and Theorem 4.4,

$$\int_{G\setminus U} |P_n|(dx) \le C\big|\langle P_n, 1-\phi\rangle\big| \to C\big|\langle P_0, 1-\phi\rangle\big| \le C\int_{G\setminus V} |P_0|(dx) < C\varepsilon,$$

which implies the first part of the claim. The second one follows by the first and Theorem 4.4 again. □

Corollary 4.10 *Let $\mu_n(t)$ and P_n satisfy the hypothesis of Theorem 4.4. Then, for every bounded $f \in C^2(G)$,*

$$\langle P_n, f\rangle \to \langle P_0, f\rangle.$$

Proof Let $\phi \in C_c^\infty(G)$ be equal to 1 in a neighbourhood of e. If $f \in C^2(G)$ is bounded, then

$$f = \phi f + (1-\phi) f,$$

where $\phi f \in C_c^2(G)$ and $(1-\phi) f$ is supported away from the identity. Our claim follows by Theorem 4.4 and Corollary 4.9. □

5 Representations

Let $\mu(t)$ be a continuous semigroup of measures on G with the generating functional P. Let π be a bounded strongly continuous representation of G on a Banach space E. The operators

$$\pi_{\mu(t)}u = \int \pi(x)\mu(t)(dx), \quad u \in E, \tag{5.1}$$

form a strongly continuous contraction semigroup. Denote by π_P the infinitesimal generator and by $\operatorname{dom}\pi_P$ the domain of π_P.

Proposition 5.2 (Duflo [2], Sects. 7, 11, and 12) *The domain* $\operatorname{dom}\pi_P$ *consists of all vectors* $u \in E$ *for which there exists a vector* $u_0 \in E$ *such that*

$$\pi_f u_0 = \pi_{f\star P}u, \tag{5.3}$$

for all $f \in C_c^\infty(G)$. *Then,* $u_0 = \pi_P u$. *Moreover, the Gårding space*

$$E^g(\pi) = \left\{\pi_f u : u \in E,\, f \in C_c^\infty(G)\right\}$$

is a core domain for π_P.

Let us denote by $C^2(E,\pi)$ the subspace of $u \in E$ such that the vector-valued function

$$G \ni x \mapsto \pi(x)u \in E$$

is twice continuously differentiable and bounded. Note that no boundedness of the derivatives is assumed.

Remark 5.4 If $\pi_y f(y) = f(xy)$ is the right-regular representation of G on the Banach space $E = \mathcal{C}(G)$, then

$$\pi_{\mu(t)} f = f \star \mu(t)^{\#}$$

so the semigroup acts on the right and does not commute with the leftinvariant derivatives. Thus, the situation is somewhat different from that in the proof of Theorem 4.1. In that respect, the reader may wish to compare Hunt [9] (action on the right) and Hulanicki [8] (action on the left).

Corollary 5.5 *If* $u \in C^2(E,\pi) \cap \operatorname{dom}\pi_P$, *then, for every* $v \in E'$

$$\langle \pi_P u, v\rangle = \langle P, f_{u,v}\rangle,$$

where $f_{u,v}(x) = \langle \pi(x)u, v\rangle$ *for* $x \in G$.

Proof In fact, let $u \in C^2(E,\pi) \cap \operatorname{dom}\pi_P$. Let f_n be an approximate identity in $C_c^\infty(G)$. Then, by Proposition 5.2, for every $v \in E'$,

$$\langle \pi_P u, v\rangle = \lim_{n\to\infty} \langle \pi_{f_n}\pi_P u, v\rangle = \lim_{n\to\infty} \langle \pi_{f_n\star P} u, v\rangle$$

$$= \lim_{n\to\infty} \langle f_n \star P, f_{u,v} \rangle = \lim_{n\to\infty} \langle P, \widetilde{f_n} \star f_{u,v} \rangle = \langle P, f_{u,v} \rangle$$

since $\widetilde{f_n} \star f_{u,v} \to f_{u,v}$ C^2-almost uniformly and boundedly. □

Remark 5.6 As a matter of fact, $C^2(E,\pi) \subset \operatorname{dom} \pi_P$, as is shown in Proposition 5.7 below.

Let X be a leftinvariant vector field on G. Then, the distribution $X(e)$ (see Sect. 2) is dissipative and the continuous semigroup of measures generated by $X(e)$ is $\mu(t) = \delta_{\exp tX}$.

The following proposition seems classical but we do not know any reference and, therefore, include a proof sketch.

Proposition 5.7 *Let P be a dissipative distribution on G. Then, $C^2(E,\pi)$ is contained in the domain of π_P. Thus,*

$$E^g(\pi) \subset C^2(E,\pi) \subset \operatorname{dom} \pi_P \subset E.$$

Proof Let $U \subset \bar{U} \subset V$ be open relatively compact neighbourhoods of the identity. Being dissipative P can be represented as $P = D + Q + \eta_1$, where D is a dissipative distribution supported at the identity, η_1 is a bounded measure supported in $G \setminus U$, and

$$\langle Q, f \rangle = \int_{V\setminus\{e\}} \left(f(x) - f(e) - \sum_{k=1}^{d} X_k f(e) \Phi_k(x) \right) Q(dx)$$

has compact support contained in V (cf., e.g., Faraut [4], below Proposition II.2). The operator π_{η_1} is bounded and the case of π_D is easy so we concentrate on π_Q.

Let $u \in C^2(E,\pi)$. Let $v \in E'$ and let $f_{u,v}(x) = \langle \pi(x)u, v \rangle$. By hypothesis, $f_{u,v} \in C_b^2(G)$. By (3.12),

$$\left| f_{u,v}(x) - f_{u,v}(e) - \sum_k X_k f_{u,v}(e) \right| \le C\Phi^2(x)\|v\|,$$

which shows that

$$\int \left\| \pi(x)u - u - \sum_k \pi_{X_k(e)} u \right\| |Q|(dx) < \infty,$$

and so the vector

$$u_0 = \int_{V\setminus\{e\}} \left(\pi(x)u - u - \sum_k \pi_{X_k(e)} u \right) Q(dx)$$

is well defined as an integral of a continuous vector-valued function with respect to the Radon measure $Q(dx)$, and, for every $v \in E'$,

$$\langle u_0, v \rangle = \langle Q, f_{u_0,v} \rangle.$$

Consequently, for every $f \in C_c^\infty(G)$,

$$\langle \pi_f u_0, v\rangle = \int f(x)\langle Q, {}_x f_{u_0,v}\rangle\, dx = \langle f, f_{u_0,v} \star Q^\#\rangle$$
$$= \langle f \star Q, f_{u_0,v}\rangle = \langle \pi_{f\star Q} u_0, v\rangle,$$

which shows that $\pi_f u_0 = \pi_{f\star Q} u$. This, by Theorem 5.2, completes the proof. □

Corollary 5.8 *Let π be a bounded strongly continuous representation of G on a Banach space E. If $\mu_n(t)$ and P_n satisfy the hypothesis of Theorem* 4.4, *then for every $u \in C^2(E,\pi)$ and every $v \in E'$,*

$$\langle \pi_{P_n} u, v\rangle \underset{n}{\to} \langle \pi_{P_0} u, v\rangle.$$

Proof If $u \in C^2(E,\pi)$ and $v \in E'$, then $f_{u,v}$ is a bounded function in $C^2(G)$ so, by Corollary 4.10, Proposition 5.7, and Corollary 5.5,

$$\langle \pi_{P_n} u, v\rangle = \langle P_n, f_{u,v}\rangle \underset{n}{\to} \langle P_0, f_{u,v}\rangle = \langle \pi_{P_0} u, v\rangle. \tag{5.9}$$

□

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