Hunt's Formula for $\mathrm{SU}_q(N)$ *and* $\mathrm{U}_q(N)$

UWE FRANZ, ANNA KULA, J. MARTIN LINDSAY & MICHAEL SKEIDE

ABSTRACT. For any Lévy process on the quantum group $\mathrm{SU}_q(N)$, where $0 < q < 1$ and $N \in \mathbb{N}$, a Lévy-Khintchine-type decomposition of its generating functional is given, together with an analogue of Hunt's formula. The non-Gaussian component is shown to further decompose into generating functionals that live on the quantum subgroups $\mathrm{SU}_q(n)$, for $n \leqslant N$. Corresponding results are also given for the quantum groups $\mathrm{U}_q(N)$.

1. INTRODUCTION

Up to stochastic equivalence, a Lévy process with values in a locally compact Lie group G is determined by its generating functional. This is a (densely defined) linear functional γ on $C_0(G)$, the C^*-algebra of continuous complex-valued functions on G which vanish at infinity, whose domain may be thought of as consisting of those functions that have a second-order Taylor expansion around the identity element of the group. Hunt's formula ([11]) is a generalization and extension of the Lévy-Khintchine formula ([1], [18]). It is equivalent to the assertion that

$$\gamma = \gamma_D + \gamma_G + \gamma_L \quad \text{where } \gamma_L = L \circ P \text{ and } L(f) = \int_{G\setminus\{e\}} f(s)\Pi(\mathrm{d}s) \tag{1.1}$$

for the identity element e of G, in which P is a Hermitian projection that kills the linear terms, the drift γ_D and P-invariant Gaussian part γ_G are linear combinations of first- and second-order derivatives evaluated at e respectively, and Π is the so-called Lévy measure. The Lévy functional L is defined on the space of functions that, together with their first derivatives, vanish at e. The integral may be viewed as a mixture of point evaluations; moreover, functionals of the form $f \mapsto f(s) - f(e)$, for fixed $s \neq e$, generate jump processes. The functional γ_L

Indiana University Mathematics Journal ©, Vol. 72, No. 4 (2023)

is also referred to as the jump part; in the case where $G = \mathbb{R}$ and Π is finite, it generates a compound Poisson process. The decomposition depends on the non-canonical projection P chosen; its role is to deal with any singularity of the measure Π at e.

If G is compact, Tannaka-Krein duality ([10, Section VII.30]) asserts that the representative algebra $R(G)$, generated by matrix coefficients of finite-dimensional representations of G, is a norm-dense $*$-subalgebra of the unital C^*-algebra $C(G)$. In fact, $R(G)$ is a commutative Hopf $*$-algebra from which the topological group G may be fully recovered ([16]). A compact quantum group in the sense of Woronowicz ([29]) is a unital C^*-algebra-with-coproduct which enjoys density relations corresponding to the group cancellation law and contains a dense Hopf $*$-algebra, the CQG algebra of the quantum group, whose role corresponds to that played by $R(G)$ for a compact group G ([4]). Schürmann's theory of quantum Lévy processes on $*$-bialgebras ([20]) thereby applies. As with their classical counterparts, but now up to *quantum* stochastic equivalence, Lévy processes on $*$-bialgebras are classified by their generating functional, now a Hermitian linear functional on the CQG algebra which is conditionally positive and vanishes at the identity element. The problem of finding a decomposition of generating functionals corresponding to (1.1) is expressible in cohomological terms. Of course, meaning has to be given to *drift*, *Gaussian*, and *jump* parts in the quantum generalisation. Our Hunt formula includes an explicit description of the drifts and Gaussian generating functionals, and the specification of an approximation property that justifies calling the remainder a jump part (Proposition 2.8).

For some compact quantum groups every generating functional has such a decomposition, but for others that is not so ([8], [2]). A Hunt formula for Woronowicz's $\mathrm{SU}_q(2)$ ([26], [27]) was obtained in [23], [21]. This led to a short proof of the classical Hunt formula for compact Lie groups ([24]). Here, we tackle the case of $\mathrm{SU}_q(N)$, obtaining a unique decomposition $\gamma = \gamma_D + \gamma_G + \gamma_{NG}$ where $\gamma_{NG} = \gamma_2 \circ P + \cdots + \gamma_N \circ P$, in which P is a Hermitian projection analogous to that of (1.1), γ_D is a drift, γ_G is a P-invariant Gaussian generating functional and, for $2 \leqslant n \leqslant N$, γ_n is an extension to $\mathrm{SU}_q(N)$ of a completely non-Gaussian generating functional on $\mathrm{SU}_q(n)$ which enjoys an irreducibility property. We also display the essentially classical structure of γ_D and γ_G, and show γ_{NG} to be the limit of functionals of the form $\omega_{\xi(t)} \circ \pi \circ P$ for a representation π and net of vector functionals $(\omega_{\xi(t)})$ (Theorem 4.15). The case of general N turns out to be more involved than the case $N = 2$, and some results concerning $\mathrm{SU}_q(2)$ fail for $N \geqslant 3$. For instance, for $N \geqslant 3$ the cohomological problem is not always solvable in the Gaussian case (Corollary 2.13). Also, for $N = 2$ the completely non-Gaussian generating functionals may be parametrized by the vectors in its associated representation Hilbert space, whereas for $N \geqslant 3$ the situation is more subtle (Section 5).

The paper is organized as follows. Terminology and notation concerning the CQG algebra of a compact quantum group are set out below. Section 2 contains the basic definitions and preliminary results. The CQG algebras of the compact quantum groups $\mathrm{SU}_q(N)$ and $\mathrm{U}_q(N)$ are, respectively, here denoted $\mathcal{SU}_q(N)$ and $\mathcal{U}_q(N)$; the former is algebraically generated by a matrix of elements $[u_{jk}]_{j,k=1}^N$ (see relations (2.6), et seq.). In Section 3 we deal with our choice of projection P, with respect to which we show that the Gaussian generating functionals on $\mathcal{SU}_q(N)$ are classified by a real $(N-1)$-vector and positive-definite real $(N-1)\times(N-1)$ matrix, respectively, representing the drift and P-invariant diffusion-type second-order term (Theorem 3.6). Unlike in lower dimensions, for $N \geqslant 3$ there are cocycles of Gaussian representations which have no associated generating functionals (Theorem 3.3). Every Gaussian generating functional is induced from a Gaussian generating functional that lives on the classical undeformed subgroup $\mathbb{T}^{N-1}$ of $\mathrm{SU}_q(N)$, in the sense of Definition 2.21 (see Remark 3.7). In Section 4 we show that every representation π of $\mathcal{SU}_q(N)$ has a unique *full* decomposition $\pi_1 \oplus \cdots \oplus \pi_N$, where the representation π_1 is its so-called Gaussian part and, for $2 \leqslant n \leqslant N$, the representation π_n lives on $\mathcal{SU}_q(n)$ and $\pi_n(1-u_{nn})$ is injective. Completely non-Gaussian cocycles η are approximated by coboundaries and are determined by their values $\eta(u_{nn})$ $(2 \leqslant n \leqslant N)$. From this we deduce a full decomposition $\gamma = \gamma_1 + \cdots + \gamma_N$ for generating functionals, uniquely determined by the projection P, and conclude with our Hunt formula (Theorem 4.15). In Section 5 we show that, unlike in the case $N=2$, if $N>2$ then the values of $\eta(u_{NN})$, for cocycles η of representations π for which $\pi(1-u_{NN})$ is injective, may not exhaust the representation space. We then indicate a completion process which yields a quasi-innerness property, and thereby full parametrisation, for completely non-Gaussian cocycles. In Section 6 we briefly treat the quantum groups $\mathrm{U}_q(N)$.

Our work suggests the investigation of Hunt formulae for other q-deformed compact Lie groups ([16]).

Compact quantum groups and CQG algebras. A *CQG algebra* ([4]), or algebraic compact quantum group, is a Hopf *-algebra $\mathcal{G}$ that is linearly spanned by the coefficients of its finite-dimensional unitary corepresentations or, equivalently, has a faithful Haar state. Thus, a CQG algebra is a unital *-algebra $\mathcal{G}$, with unital *-algebra morphisms $\Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ and $\varepsilon : \mathcal{G} \to \mathbb{C}$, linear map $\kappa : \mathcal{G} \to \mathcal{G}$, and unital linear functional $h : \mathcal{G} \to \mathbb{C}$, called respectively, the coproduct, counit, coinverse or antipode, and Haar state, enjoying the coassociativity, counital, coinverse, invariance, and positivity relations

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta;\\
&(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta;\\
&\mu \circ (\mathrm{id} \otimes \kappa) \circ \Delta = \iota \circ \varepsilon = \mu \circ (\kappa \otimes \mathrm{id}) \circ \Delta;\\
&(\mathrm{id} \otimes h) \circ \Delta = \iota \circ h = (h \otimes \mathrm{id}) \circ \Delta;
\end{aligned}$$

and $h(a^*a) > 0$ for $a \neq 0$. Here, $\mu : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ denotes the linearisation of the algebra product, and ι the unital linear map $\mathbb{C} \to \mathcal{G}$. The coinverse κ is uniquely determined by the bialgebra structure, and any $*$-bialgebra morphism between CQG algebras respects coinverses and so is a CQG algebra morphism (Remarks 4.2.3 and 4.2.5 in [3]); the Haar state h is also unique ([4, Proposition 3.2]). Compact quantum groups may also be viewed from the equivalent C^*-algebraic perspective, as was originally done by Woronowicz ([29]). The canonical (universal and reduced) Woronowicz algebras of a compact quantum group $\mathbb{G}$ are commonly denoted $C_{\mathrm{u}}(\mathbb{G})$ and $C_{\mathrm{r}}(\mathbb{G})$, and its CQG algebra is here denoted by $R(\mathbb{G})$ in a further nod to their classical counterparts. The quantum space $\mathbb{G}$ itself is only manifested through one of its realisations. For more on this, we recommend [16], [12, Section 11.3], and [25, Section 5.4]. For the purposes of this work, it suffices to operate exclusively within CQG algebras. In fact, in our analysis we need *explicit* recourse to none of the coproduct, coinverse, or Haar state.[1]

Convention. In Schürmann's theory, representations are by possibly-unbounded adjointable operators on pre-Hilbert spaces because he works in the more general setting of $*$-bialgebras-with-character. By contrast, representations of a CQG algebra $\mathcal{G}$ are all by bounded operators, and so may be extended to the Hilbert space completions. Accordingly, by a *representation of* $\mathcal{G}$ we always mean a unital $*$-algebra morphism $\pi : \mathcal{G} \to B(\mathsf{h})$, for some Hilbert space $\mathsf{h} = \mathsf{h}^\pi$.

2. PRELIMINARIES

Generating functionals of quantum Lévy processes and Schürmann triples. Let $\mathcal{G}$ be a CQG algebra. A *Lévy process on* $\mathcal{G}$ is a family of $*$-algebra morphisms from $\mathcal{G}$ to a noncommutative probability space enjoying certain properties which encode the stationarity and independence of increments (see [20], [5], and [15, Chapter VII], or the survey [6]).

Definition 2.1. A *generating functional* for a quantum Lévy process on $\mathcal{G}$ is a linear functional γ on $\mathcal{G}$ which is *Hermitian*: $\gamma = \gamma^\dagger : a \mapsto \overline{\gamma(a^*)}$, *normalised*: $\gamma(1) = 0$, and *conditionally positive*: $\gamma(c^*c) \geqslant 0$ for all $c \in \ker \varepsilon$.

Quantum Lévy processes are determined up to quantum stochastic equivalence by their generating functionals, and may be reconstructed from their generating functional using quantum stochastic calculus on a symmetric Fock space ([20, Theorem 2.3.5], [14, Theorem 7.1]), or using Trotter products and Arveson (product) systems ([22]).

Definition 2.2 ([20]). A *Schürmann triple* on $\mathcal{G}$ is an ordered triple (π, η, γ) consisting of a representation π of $\mathcal{G}$, a π-ε-cocycle, or π-ε-derivation, that is, a linear mapping $\eta : \mathcal{G} \to \mathsf{h}^\pi$ satisfying

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b) \quad (a, b \in \mathcal{G}), \tag{2.1}$$

[1] The fourth author wishes to emphasise that revisions for this final version of the paper were done by the other authors, and that the original version is available on the arXiv ([9]).

and a linear functional γ on $\mathcal{G}$ satisfying

$$\gamma^\dagger = \gamma,\ \gamma(1) = 0 \text{ and } \gamma(c^*c) = \|\eta(c)\|^2 \quad (c \in \ker\varepsilon), \tag{2.2}$$

equivalently, $\gamma^\dagger = \gamma$ and $\langle \eta(a), \eta(b)\rangle = \gamma(a^*b) - \overline{\gamma(a)}\varepsilon(b) - \overline{\varepsilon(a)}\gamma(b)$ for $a, b \in \mathcal{G}$.

A linear functional γ on $\mathcal{G}$ *completes* a π-ε-cocycle η if (π, η, γ) is a Schürmann triple; we then say that η, or (π, η), is *completable*.

A Schürmann triple (π, η, γ) or cocycle η, is called *cyclic* if $\overline{\eta(\mathcal{G})} = \mathsf{h}^\pi$.

The third component of a Schürmann triple is a generating functional. Conversely, for any generating functional γ, there is a cyclic Schürmann triple with γ as its third component. If (π, η, γ) is a cyclic Schürmann triple then, for any linear isometry V from h^π into a Hilbert space, $(V\pi(\cdot)V^*, V\eta(\cdot), \gamma)$ is a Schürmann triple (cyclic if and only if V is unitary), and every Schürmann triple having γ as its third component is of this form. Thus, all cyclic Schürmann triples having γ as their third component are unitarily equivalent—we refer to any one of these as γ's (*associated*) *Schürmann triple* ([20, Section 2.3]).

For $K := \ker\varepsilon$, set

$$K_n := \operatorname{span}\{c_1 \cdots c_n : c_1, \ldots, c_n \in K\} \text{ for } n \geqslant 1 \quad \text{and} \quad K_\infty := \bigcap_{n\geqslant 1} K_n.$$

Thus, (K_n) is a sequence of $*$-ideals of $\mathcal{G}$ decreasing to K_∞. Also set

$$P_2(\mathcal{G}) := \{P \in L(\mathcal{G}) : P \text{ is a Hermitian projection, } \operatorname{ran} P = K_2 \text{ and } 1 \in \ker P\},$$

where *Hermitian* means $P = P^\dagger : a \mapsto (Pa^*)^*$ for $a \in \mathcal{G}$.

Definition 2.3. Let γ be a generating functional on $\mathcal{G}$. Then, γ is a *drift* if $\gamma|_{K_2} = 0$, equivalently, in terms of its associated Schürmann triple (π, η, γ), if $\mathsf{h}^\pi = \{0\}$.

For $P \in P_2(\mathcal{G})$, we denote the drift $\gamma - \gamma \circ P$ by γ_D^P, and call γ *P-invariant* if $\gamma \circ P = \gamma$.

Remarks 2.4. The drifts form a real subspace of the linear dual of $\mathcal{G}$. Any $P \in P_2(\mathcal{G})$ determines a unique resolution for generating functionals γ into a drift component plus a P-invariant one: $\gamma = \gamma_D^P + \gamma \circ P$—in this sense P-invariance may usefully be thought of as a *P-driftless* property (i.e., having zero drift component with respect to P). If a cocycle η is completable then, for any particular generating functional γ which completes η, the set of all generating functionals which do so equals $\{\gamma + \gamma' : \gamma' \text{ is a drift}\}$ and the unique P-invariant one is $\gamma \circ P = \gamma - \gamma_D^P$.

The P-invariant generating functionals on $\mathcal{G}$ are the maps of the form $\psi \circ P$ for a linear functional ψ on K_2 which is nonnegative: $\psi(c^*c) \geqslant 0$ for all $c \in K$ (and thus also Hermitian).

There is no canonical choice of projection from $P_2(\mathcal{G})$. By contrast, since $\mathbb{C}1$ and K are complementary subspaces of $\mathcal{G}$, there is a unique projection in $L(\mathcal{G})$

with range K and 1 in its kernel—namely, $\mathrm{id} - \iota \circ \varepsilon : a \mapsto a - \varepsilon(a)1$. Moreover, it is Hermitian and compatible with the projections in $P_2(\mathcal{G})$.

Definition 2.5. Let U be a subspace of a complex vector space V. A linearly independent subset E of $V \setminus U$ is a *basis extension from U to V* if its linear span is a complementary subspace of U. In case V is involutive, a basis extension is *Hermitian* if it consists of self-adjoint elements.

For any Hermitian basis extension E from K_2 to K, the functionals $(\varepsilon'_d)_{d\in E}$ on $\mathcal{G}$ given by

$$\varepsilon'_d\Big(\lambda 1 + k_2 + \sum_{e\in E} \lambda_e e\Big) = \lambda_d, \qquad \text{for } \lambda \in \mathbb{C},\ k_2 \in K_2 \text{ and } \{\lambda_e : e \in E\} \subset \mathbb{C}, \tag{2.3}$$

form a basis for the real space of drifts on $\mathcal{G}$, and

$$P^E := \mathrm{id} - \iota \circ \varepsilon - \sum_{d\in E} d\varepsilon'_d(\cdot) \in P_2(\mathcal{G}) \tag{2.4}$$

equals the projection onto K_2 along $\mathrm{span}(\{1\} \cup E) = \mathbb{C}1 \oplus \mathrm{span}\, E$. The resulting map $E \mapsto P^E$ is surjective and $P^{E_1} = P^{E_2}$ if and only if $\mathrm{span}\, E_1 = \mathrm{span}\, E_2$.

Procedure 2.6. All generating functionals on $\mathcal{G}$ are obtained by identifying the following:

(1) the representations π of $\mathcal{G}$;
(2) for each representation π, the π-ε-cocycles η;
(3) for each such cocycle η, the generating functionals γ which complete it.

In the cases of the quantum groups $\mathrm{SU}_q(N)$ and $\mathrm{U}_q(N)$, the representation theory is known ([13]). Step (2) is a cohomological problem, as π-ε-cocycles form the first Hochschild cohomology group $H^1(\mathcal{G}, {}_\pi\mathsf{h}_\varepsilon)$ for $\mathsf{h} = \mathsf{h}^\pi$, and this may usually be computed in a straightforward way. The main problem lies in step (3). The basic constraint on a given cocycle η, for it to be completable, is that $\|\eta(c)\|$ must equal $\|\eta(d)\|$ whenever $c, d \in K$ satisfy $c^*c = d^*d$; the task then amounts to solving $\psi(c^*c) = \|\eta(c)\|^2$ $(c \in K)$ for a linear functional ψ on K_2 since then, for any $P \in P_2(\mathcal{G})$, the prescription $a \mapsto \psi(Pa)$ defines a (P-invariant) generating functional which completes η.

Approximately inner cocycles. As just described, the problem of classifying generating functionals on $\mathcal{G}$ lies in the fact that there might be none which completes a given cocycle. In this section we identify a situation where such a completion does exist.

Definition 2.7. A π-ε-cocycle is a *coboundary*, or *inner derivation*, if it is of the form

$$\eta_{\pi,\xi} := (\pi - \iota \circ \varepsilon)(\cdot)\xi : a \mapsto \pi(a)\xi - \xi\varepsilon(a)$$

for some vector ξ in h^π, and is *approximately inner* if it is a pointwise limit of coboundaries $(\eta_{\pi,\xi(\lambda)})$ for some net $(\xi(\lambda))$ in h^π.

Note that, for a vector ξ of a Hilbert space h, ω_ξ denotes the vector functional $T \mapsto \langle \xi, T\xi \rangle$ on $B(\mathsf{h})$. The following result is heavily used in Section 4.

Proposition 2.8. *Approximately inner cocycles are completable. Specifically, let* $P \in P_2(\mathcal{G})$*, let* π *be a representation of* $\mathcal{G}$*, and let* $(\xi(\lambda))$ *be a net in* h^π *such that* $(\eta_\lambda := \eta_{\pi,\xi(\lambda)})$ *converges pointwise to a map* η*. Then,* η *is a* π*-*ε*-cocycle and the net* $(\gamma_\lambda := \omega_{\xi(\lambda)} \circ \pi \circ P)$ *converges pointwise to a* P*-invariant generating functional* γ *which completes* η*.*

Proof. For each λ, the P-invariant linear functional γ_λ is Hermitian, and $(\pi, \eta_\lambda, \gamma_\lambda)$ is easily seen to satisfy (2.1) and (2.2). Therefore, since η is evidently a π-ε-cocycle and K_2 is both the range of P and the linear span of the set $\{c^*c : c \in K\}$, the proposition follows from the fact that $\gamma_\lambda(c^*c) = \|\pi(c)\xi(\lambda)\|^2 = \|\eta_{\pi,\xi(\lambda)}(c)\|^2 \to \|\eta(c)\|^2$ for each $c \in K$. ◻

In the classical setting of (1.1), we see that the generating functional γ_L is expressible as the limit of the functionals $\omega_{1_{G\setminus U}} \circ \pi \circ P$, as the neighbourhoods U of e shrink to $\{e\}$, π being the multiplication representation of $R(G)$ on $L^2(G, \Pi)$ and 1 here denoting indicator function.

Gaussian generating functionals, cocycles, and representations.

Definition 2.9. A generating functional γ, cocycle η, or representation π is called *Gaussian* if it vanishes on, respectively, K_3, K_2, or K.

For components of a Schürmann triple, these are equivalent (Proposition 5.1.1 in [20]). A representation π is Gaussian if and only if $\pi = \iota_{\mathsf{h}^\pi} \circ \varepsilon$, where ι_{h^π} denotes the unital linear map from $\mathbb{C}$ to $B(\mathsf{h}^\pi)$.

Proposition 2.10. *Let* E *be a Hermitian basis extension from* K_2 *to* K*. Then, for any Hilbert space* h*, the* h*-valued Gaussian cocycles on* $\mathcal{G}$ *are precisely the maps of the form* $\sum_{d \in E} \xi_d \varepsilon'_d(\cdot)$ *for a family of vectors* $(\xi_d)_{d \in E}$ *in* h*, where the functionals* ε'_d *are as in* (2.3)*.*

Proof. Since Gaussian cocycles vanish on 1 and on K_2, this follows from the fact that elements a of $\mathcal{G}$ are uniquely expressible as $\varepsilon(a)1 + k_2(a) + \sum_{d \in E} \varepsilon'_d(a)d$ for some $k_2(a) \in K_2$. ◻

It would be desirable to have a similarly concise description of Gaussian generating functionals. For now, we note that in general not all Gaussian cocycles η admit a Gaussian generating functional.

Definition 2.11. A cocycle η on $\mathcal{G}$ is *Hermitian* if it satisfies $\|\eta(c)\| = \|\eta(c^*)\|$ for all $c \in K$.

A Gaussian cocycle of the form $\eta = \sum_{d \in E} \xi_d \varepsilon'_d$ is Hermitian if and only if the Gram matrix $[\langle \xi_d, \xi_{d'} \rangle]$ is real (and therefore symmetric). Proposition 2.10 has the following consequence.

Corollary 2.12. *The following are equivalent:*

(i) $\mathcal{G}$ *has non-Hermitian Gaussian cocycles.*

(ii) $\dim K/K_2 \geqslant 2$.

For a Gaussian cocycle η to be completable it is sufficient, but not necessary, that it be Hermitian ([20, Proposition 5.1.11]). It becomes necessary too under the additional assumption given in the next corollary, which applies to both $\mathrm{SU}_q(N)$ (by Lemma 3.2 and part (d) of Lemma 3.1), and $\mathrm{U}_q(N)$.

Corollary 2.13. *Suppose $c^*c - cc^* \in K_3$ for all $c \in K$. Then, a Gaussian cocycle is completable if and only if it is Hermitian.*

Proof. If γ is a generating functional completing a Gaussian cocycle η, then η is Hermitian since

$$\|\eta(c)\|^2 - \|\eta(c^*)\|^2 = \gamma(c^*c - cc^*) = 0 \quad (c \in K).$$

The converse implication is clear. ◻

Complete non-Gaussianness and Lévy-Khintchine decomposition. We will next collect basic facts about when a generating functional can have a Lévy-Khintchine decomposition.

Lemma 2.14. *Let $\pi_1 \oplus \pi_2$ be a decomposition of a representation π of $\mathcal{G}$, let V_i denote the inclusion map $\mathsf{h}^{\pi_i} \to \mathsf{h}^{\pi}$ for $i = 1, 2$, and let η be a π-ε-cocycle. Then, the following hold:*

(a) $\eta_i := V_i^* \eta(\cdot)$ *is a π_i-ε-cocycle for $i = 1, 2$.*
(b) *If two of the three cocylces η, η_1, and η_2 are completable, then so is the third.*

It is quite possible that η is completable, but η_1 and η_2 are not.

Definition 2.15. For a representation π of $\mathcal{G}$, set

$$\mathsf{h}^{\pi_G} := \bigcap_{c \in K} \ker \pi(c) \text{ and } \mathsf{h}^{\pi_R} := (\mathsf{h}^{\pi_G})^{\perp}.$$

Then, π is *completely non-Gaussian* if $\mathsf{h}^{\pi_G} = \{0\}$, or equivalently, if $\mathsf{h}^{\pi_R} = \mathsf{h}^{\pi}$.

We also call a π-ε-cocycle η completely non-Gaussian if π is, and a generating functional γ completely non-Gaussian if the representation component of its Schürmann triple is.

The above definition and its notation are amply justified by the following straightforward proposition.

Proposition 2.16 ([20]). *Let π be a representation of $\mathcal{G}$. Then, h^{π_G} and h^{π_R} are invariant subspaces and, denoting the resulting decomposition of π as $\pi_G \oplus \pi_R$, π_G is Gaussian and π_R is completely non-Gaussian. Moreover, $\mathsf{h}^{(\pi_R)_G} = \{0\} = \mathsf{h}^{(\pi_G)_R}$.*

If $\eta = \eta_G \oplus \eta_R$ is the corresponding decomposition of a π-ε-cocycle η then η_G is Gaussian, and if η is cyclic then η_G and η_R are cyclic too.

Generating functionals of the form $\omega_\xi \circ \pi \circ P$, and their limits as in Proposition 2.8, are completely non-Gaussian.

Definition 2.17. A *Lévy-Khintchine decomposition* for a generating functional γ with Schürmann triple (π, η, γ) is a decomposition $\gamma = \gamma_1 + \gamma_2$ for which

$(\pi_G, \eta_G, \gamma_1)$ and $(\pi_R, \eta_R, \gamma_2)$ are Schürmann triples (equivalently, by Lemma 2.14, one of them is).

Remark 2.18. With respect to a fixed projection $P \in P_2(\mathcal{G})$, if γ has such a Lévy-Khintchine decomposition then it has a unique one in which $\gamma_1 = \gamma_D^P + \gamma_G$, $\gamma_2 = \gamma_R$, and the generating functionals γ_G and γ_R are P-invariant.

Definition 2.19. A CQG algebra, or its associated quantum group, is said to have the following *properties*:

- (AC) if each cocycle η is completable.
- (GC) if each Gaussian cocycle η is completable.
- (NC) if each completely non-Gaussian cocycle η is completable.
- (NAI) if each completely non-Gaussian cocycle η is approximately inner.
- (LK) if every generating functional admits a Lévy-Khintchine decomposition.

Evidently, (AC) implies both (GC) and (NC), and either of these implies (LK); none of the reverse implications hold ([8]). The following is an immediate consequence of Proposition 2.8.

Proposition 2.20. *Property* (NAI) *implies property* (NC)*, and thus also property* (LK).

Schürmann triples on quantum subgroups. In the course of proving our results for $\mathrm{SU}_q(N)$, we will decompose representations into components that live on its quantum subgroups $\mathrm{SU}_q(n)$ in the sense given below. One way of extending our results to $\mathrm{U}_q(N)$ is by exploiting the quantum subgroup relations $\mathbb{T}^N \leqslant \mathrm{U}_q(N) \leqslant \mathrm{SU}_q(N+1)$; this is done in Section 6.

Definition 2.21. A compact quantum group $\mathbb{H}$ is a *quantum subgroup* of a compact quantum group $\mathbb{G}$, written $\mathbb{H} \leqslant \mathbb{G}$, if there is a CQG algebra epimorphism (equivalently, a *-bialgebra epimorphism) $s\colon \mathcal{G} \to \mathcal{H}$; we also say that $(\mathcal{H}, s)$ is a quantum subgroup of $\mathcal{G}$.

Given such a subgroup relation, we say that a linear map T from $\mathcal{G}$ to a vector space V *lives on* $(\mathcal{H}, s)$ if $\ker T \supset \ker s$, equivalently, if T factors (evidently uniquely) through the epimorphism s:

$$T = \tilde{T} \circ s \quad \text{for some map } \tilde{T} : \mathcal{H} \to V.$$

For the remainder of this subsection we fix a quantum subgroup $(\mathcal{H}, s)$ of $\mathcal{G}$ and use tildes for induced maps having domain $\mathcal{H}$. Since s respects counits, the functional $\tilde{\varepsilon}$ on $\mathcal{H}$ satisfying $\tilde{\varepsilon} \circ s = \varepsilon$ is its counit, and $s(K_n) = \tilde{K}_n$ for all n. Also, a representation of $\mathcal{G}$ lives on the trivial CQG algebra $\mathbb{C}$ if and only if it is Gaussian. The properties listed next are easily verified.

Lemma 2.22. *Suppose that* $\pi = \tilde{\pi} \circ s$*,* $\eta = \tilde{\eta} \circ s$ *and* $\gamma = \tilde{\gamma} \circ s$*, for maps* $\pi, \dots, \tilde{\gamma}$*; then, the following hold:*

1. π *is a representation of* $\mathcal{G}$ *if and only if* $\tilde{\pi}$ *is a representation of* $\mathcal{H}$.
2. *If* (1) *holds then* η *is a* π-ε*-cocycle if and only if* $\tilde{\eta}$ *is a* $\tilde{\pi}$-$\tilde{\varepsilon}$*-cocycle.*

(3) *γ is a generating functional on $\mathcal{G}$ if and only if $\tilde{\gamma}$ is a generating functional on $\mathcal{H}$.*
(4) *(π, η, γ) is a Schürmann triple on $\mathcal{G}$ if and only if $(\tilde{\pi}, \tilde{\eta}, \tilde{\gamma})$ is a Schürmann triple on $\mathcal{H}$.*

Moreover, for any representation π of $\mathcal{G}$ living on $(\mathcal{H}, s)$ and vector ξ in h^π,

$$\mathsf{h}^{\widetilde{\pi_G}} = \mathsf{h}^{\pi_G}, \ \eta_{\pi,\xi} \text{ lives on } \mathcal{H} \text{ and } \widetilde{\eta_{\pi,\xi}} = \eta_{\tilde{\pi},\xi}. \tag{2.5}$$

This has the following useful corollary.

Proposition 2.23. *The property* (NAI) *is hereditary.*

We now show that an approximately inner cocycle lives on a subgroup if its approximating inner cocycles do.

Proposition 2.24. *Let π be a representation of $\mathcal{G}$ living on $(\mathcal{H}, s)$, let $(\xi(\lambda))$ be a net in h^π such that $(\eta_{\pi,\xi(\lambda)})$ converges pointwise to η, and let $P' \in P_2(\mathcal{H})$. Then, the following hold:*

(a) *$(\eta_{\tilde{\pi},\xi(\lambda)})$, $(\omega_{\xi(\lambda)} \circ \pi \circ P)$, and $(\omega_{\xi(\lambda)} \circ \tilde{\pi} \circ P')$ have pointwise limits $\tilde{\eta}$, γ and γ', such that $\eta = \tilde{\eta} \circ s$, and γ and γ' are generating functionals completing η and $\tilde{\eta}$ respectively.*
(b) *$\gamma = \gamma' \circ s \circ P$.*

Proof. (a) It follows from identity (2.5) that $\eta_{\tilde{\pi},\xi(\lambda)} \circ s = \eta_{\pi,\xi(\lambda)}$ for each λ, and so (a) follows from the surjectivity of s and Proposition 2.8.

(b) This follows since $s(K_2) = \tilde{K}_2 = \operatorname{ran} P'$ so $P' \circ s \circ P = s \circ P$, and thus, for each λ, $(\omega_{\xi(\lambda)} \circ \tilde{\pi} \circ P') \circ (s \circ P) = \omega_{\xi(\lambda)} \circ \tilde{\pi} \circ s \circ P = \omega_{\xi(\lambda)} \circ \pi \circ P$. □

The projections $P \in P_2(\mathcal{G})$ and $P' \in P_2(\mathcal{H})$ may be chosen to be compatible. This follows from the following straightforward lemma.

Lemma 2.25. *Let $P = P^E$ and $P' = P^{E'}$ for Hermitian basis extensions E from K_2 to K and E' from $\tilde{K}_2$ to $\tilde{K}$, according to (2.4). Then, $P' \circ s = s \circ P$ if and only if $s(E) \subset \operatorname{span} E'$, in which case $\operatorname{span} s(E) = \operatorname{span} E'$, and so the generating functional γ from Proposition 2.24 lives on $\mathcal{H}$.*

***The quantum groups* $\mathrm{SU}_q(N)$ *and* $\mathrm{U}_q(N)$.** Let $0 < q < 1$. We next collect the facts about $\mathrm{SU}_q(N)$ and $\mathrm{U}_q(N)$ for $N \geqslant 2$ that are required. For convenience, we extend our definitions to the case $N = 1$: $\mathrm{SU}_q(1) = \mathrm{SU}(1) := \{e\}$, the trivial group, and $\mathrm{U}_q(1) := \mathrm{U}(1) = \mathbb{T}$, the torus. For an element σ of the permutation group S_N, denote the number of inversions of σ as follows:

$$i(\sigma) := \#\{(j,k) \colon j < k, \ \sigma(j) > \sigma(k)\}.$$

As a *unital algebra*, the CQG algebra $\mathcal{U}_q(N)$ of the compact quantum group $\mathrm{U}_q(N)$, is generated by indeterminates u_{jk} $(j,k = 1, \dots, N)$ and D^{-1}, subject to the following relations ([13, Section 2]):

$$u_{ij}u_{kj} = q u_{kj}u_{ij} \qquad \text{if } i < k, \tag{2.6a}$$

$$u_{ij}u_{il} = qu_{il}u_{ij} \qquad \text{if } j < l, \tag{2.6b}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} \qquad \text{if } i < k,\ j > l, \tag{2.6c}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} - (q^{-1} - q)u_{il}u_{kj} \quad \text{if } i < k,\ j < l, \tag{2.6d}$$

and

$$D^{-1}D_q = 1 = D_q D^{-1},$$

for the q-determinant of the matrix $U = [u_{jk}]_{j,k=1}^{n}$,

$$D_q = D_q(U) := \sum_{\sigma \in S_N} (-q)^{i(\sigma)} u_{1,\sigma(1)} \cdots u_{N,\sigma(N)}.$$

The jk-th q-minor is defined as the q-determinant of the $(N-1)\times(N-1)$-matrix obtained from U by removing the j-th row and the k-th column,

$$\begin{aligned} D_q^{jk} &= D_q^{jk}(U) \\ &:= \sum_{\sigma \in S_{N-1}^{jk}} (-q)^{i(\sigma)} u_{1,\sigma(1)} \cdots u_{j-1,\sigma(j-1)} u_{j+1,\sigma(j+1)} \cdots u_{N,\sigma(N)}, \end{aligned}$$

where S_{N-1}^{jk} denotes the set of bijections σ from $\{1, \ldots, j-1, j+1, \ldots, N\}$ to $\{1, \ldots, k-1, k+1, \ldots, N\}$. The involution, counit, and coproduct of $\mathcal{U}_q(N)$ are then determined by the requirements

$$\begin{aligned} u_{jk}^* &= (-q)^{k-j} D_q^{jk} D^{-1}, \\ (D^{-1})^* &= D_q, \\ \varepsilon(u_{jk}) &= \delta_{jk}, \\ \Delta u_{jk} &= \sum_{l=1}^{n} u_{jl} \otimes u_{lk}. \end{aligned}$$

The matrix of elements U satisfies the unitarity relations (2.7) below.

As a *unital *-algebra*, $\mathcal{SU}_q(N)$ is generated by indeterminates u_{jk}, $j,k = 1, \ldots, N$, subject to the unitarity relations ([28]):

$$\sum_{s=1}^{N} u_{js}u_{ks}^* = \delta_{jk},\ 1 = \sum_{s=1}^{N} u_{sj}^* u_{sk} \quad (j,k \in \{1,2,\ldots,N\}), \tag{2.7}$$

and the twisted determinant conditions

$$\sum_{\sigma \in S_N} (-q)^{i(\sigma)} u_{\sigma(1),\tau(1)} u_{\sigma(2),\tau(2)} \cdots u_{\sigma(N),\tau(N)} = (-q)^{i(\tau)} 1 \quad (\tau \in S_N).$$

The counit and coproduct are given by the same formulae as for $\mathcal{U}_q(N)$.

Remark 2.26. We also use an alternative characterisation of $S\mathcal{U}_q(N)$, namely, as the quotient of $\mathcal{U}_q(N)$ by the extra relation $D_q = 1$; the involution then simplifies to

$$u^*_{jk} := (-q)^{k-j} D_q^{jk},$$

showing that, *as an algebra*, $S\mathcal{U}_q(N)$ is generated by the u_{jk}. This means that, when checking well-definedness of representations and cocycles, one only has to manage the relations of the generators u_{jk} (namely, (2.6) and $D_q([u_{jk}]) = 1$) and not those involving their adjoints.

The following commutation relations among the generators u_{jk} of $\mathcal{U}_q(N)$ and their adjoints, and therefore also those of $S\mathcal{U}_q(N)$, are easily verified: for $i, j, k, l \in \{1, \dots, N\}$,

$$u_{ij}u^*_{kl} = u^*_{kl}u_{ij} \quad \text{if } i \neq k \text{ and } j \neq l, \tag{2.8a}$$

$$u_{ij}u^*_{kj} = qu^*_{kj}u_{ij} - (1-q^2)\sum_{m<j} u_{im}u^*_{km} \quad \text{if } i \neq k, \tag{2.8b}$$

$$u_{ij}u^*_{il} = q^{-1}u^*_{il}u_{ij} + (q^{-1}-q)\sum_{n>i} u^*_{nl}u_{nj} \quad \text{if } j \neq l, \tag{2.8c}$$

$$u_{ij}u^*_{ij} = u^*_{ij}u_{ij} + (1-q^2)\sum_{n>i} u^*_{nj}u_{nj} - (1-q^2)\sum_{m<j} u_{im}u^*_{im}. \tag{2.8d}$$

We also use the further consequences: for $1 \leqslant j, k < N$,

$$u_{Nj}u^*_{Nk} = q^{-1}u^*_{Nk}u_{Nj} \quad \text{if } j \neq k, \tag{2.9a}$$

$$u_{jN}u^*_{kN} = q^{-1}u^*_{kN}u_{jN} \quad \text{if } j \neq k, \tag{2.9b}$$

$$u^*_{NN}u_{NN} = q^2 u_{NN}u^*_{NN} + (1-q^2)1. \tag{2.9c}$$

Identity (2.9a) follows from (2.8c), identity (2.8b) with the unitarity condition (2.7) together imply that, for $j \neq k$,

$$\begin{aligned} u_{jN}u^*_{kN} &= qu^*_{kN}u_{jN} - (1-q^2)\sum_{m<N} u_{jm}u^*_{km} \\ &= qu^*_{kN}u_{jN} + (1-q^2)u_{jN}u^*_{kN}, \end{aligned}$$

from which (2.9b) follows, and identity (2.9c) follows from (2.8d):

$$\begin{aligned} u_{NN}u^*_{NN} &= u^*_{NN}u_{NN} - (1-q^2)\sum_{m<N} u_{Nm}u^*_{Nm} \\ &= u^*_{NN}u_{NN} - (1-q^2)(1 - u_{NN}u^*_{NN}). \end{aligned}$$

We next describe the relevant quantum subgroup relations. By definition, $\mathrm{SU}_q(N)$ is a quantum subgroup of $\mathrm{U}_q(N)$ via the CQG epimorphism determined by its action on generators: that is, $r_N\colon u_{jk} \longmapsto u_{jk}$ and $D^{-1} \longmapsto 1$. Also, $\mathcal{U}_q(N)$ is a quantum subgroup of $S\mathcal{U}_q(N+1)$ via the epimorphism determined by

$$t_N\colon \begin{bmatrix} u_{11} & \cdots & u_{1N} & u_{1,N+1} \\ \vdots & \ddots & \vdots & \vdots \\ u_{N1} & \cdots & u_{NN} & u_{N,N+1} \\ u_{N+1,1} & \cdots & u_{N+1,N} & u_{N+1,N+1} \end{bmatrix} \longmapsto \begin{bmatrix} u_{11} & \cdots & u_{1N} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{N1} & \cdots & u_{NN} & 0 \\ 0 & \cdots & 0 & D^{-1} \end{bmatrix}$$

where, as in the definition of r_N, the u_{jk} on the lefthand side are the generators of $S\mathcal{U}_q(N+1)$ while those on the righthand side are the generators of $\mathcal{U}_q(N)$); like r_N, t_N respects coproduct, counit and involution, and thus also coinverse. Composition gives the chain

$$\mathrm{SU}_q(1) \leqslant \mathrm{U}_q(1) \leqslant \mathrm{SU}_q(2) \leqslant \mathrm{U}_q(2) \leqslant \cdots \leqslant \mathrm{SU}_q(N) \leqslant \mathrm{U}_q(N) \leqslant \cdots .$$

Of particular interest to us is the epimorphism

$$s_N := r_{N-1} \circ t_{N-1} : S\mathcal{U}_q(N) \to S\mathcal{U}_q(N-1),$$

which is determined by

$$\text{(2.10)} \quad s_N\colon \begin{bmatrix} u_{11} & \cdots & u_{1,N-1} & u_{1N} \\ \vdots & \ddots & \vdots & \vdots \\ u_{N-1,1} & \cdots & u_{N-1,N-1} & u_{N-1,N} \\ u_{N1} & \cdots & u_{N,N-1} & u_{NN} \end{bmatrix} \longmapsto \begin{bmatrix} u_{11} & \cdots & u_{1,N-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{N-1,1} & \cdots & u_{N-1,N-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

and its iterates

$$\text{(2.11)} \quad s_{n,N} := s_{n+1} \circ \cdots \circ s_N : S\mathcal{U}_q(N) \to S\mathcal{U}_q(n) \quad (n < N).$$

Proposition 2.27. *Let* $1 \leq n < N$. *The kernel of* $s_{n,N}$ *equals the ideal* $\mathcal{I}$ *generated by the set*

$$S_{n,N} := \{u_{kj} - \delta_{kj}1 : 1 \leqslant j,k \leqslant N,\ \max\{j,k\} > n\}.$$

Proof. For $m \in \{n,N\}$ let us abbreviate $S\mathcal{U}_q(m)$ to $\mathcal{A}_m$ and denote its algebra generators by u^m_{jk} ($1 \leqslant j,k \leqslant m$). We also write $\mathcal{K}$ for the ideal $\ker s_{n,N}$ of $\mathcal{A}_N$.

For $\sigma \in S_N$ and $n < p \leqslant N$, $u^N_{p,\sigma(p)} - \delta_{p,\sigma(p)}1 \in S_{n,N} \subset \mathcal{I}$, so

$$\begin{aligned} 1 = D_q([u^N_{jk}]) &\in \sum_{\sigma\in S_N \text{ s.t. } \sigma(p)=p \text{ for } n<p\leqslant N} (-q)^{i(\sigma)} u^N_{1,\sigma(1)} \cdots u^N_{n,\sigma(n)} + \mathcal{I} \\ &= \sum_{\tau\in S_n} (-q)^{i(\tau)} u^N_{1,\tau(1)} \cdots u^N_{n,\tau(n)} + \mathcal{I} \\ &= D_q([u^N_{jk}]_{1\leqslant j,k\leqslant n}) + \mathcal{I}. \end{aligned}$$

It follows that the relation $D_q([u^n_{jk}]) = 1$ in $\mathcal{A}_n$ is preserved by the mapping from the set of generators of $\mathcal{A}_n$ into the quotient algebra $\mathcal{A}_N/\mathcal{I}$ given by $u^n_{jk} \mapsto u^N_{jk} + \mathcal{I}$ ($1 \leqslant j,k \leqslant n$). Since this clearly also preserves the (remaining defining) relations (2.6), the mapping uniquely extends to an algebra morphism $\varphi : \mathcal{A}_n \to \mathcal{A}_N/\mathcal{I}$.

Now, note that the prescription $a+\mathcal{I} \mapsto a+\mathcal{K}$ defines an algebra epimorphism $\psi : \mathcal{A}_N/\mathcal{I} \to \mathcal{A}_N/\mathcal{K}$ (since $\mathcal{I} \subset \mathcal{K}$) and, letting $\tilde{s}_{n,N}$ denote the canonically induced algebra isomorphism $\mathcal{A}_N/\mathcal{K} \to \mathcal{A}_n$,

$$(\varphi \circ \tilde{s}_{n,N} \circ \psi)(u^N_{jk} + \mathcal{I}) = (\varphi \circ \tilde{s}_{n,N})(u^N_{jk}) = \begin{cases} u^N_{jk} + \mathcal{I} & \text{if } 1 \leqslant j,k \leqslant n, \\ \delta_{jk}1 + \mathcal{I} & \text{if } \max\{j,k\} > n. \end{cases}$$

Thus, since $u^N_{jk} - \delta_{jk}1 \in S_{n,N} \subset \mathcal{I}$ if

$$\max\{j,k\} > n, \qquad (\varphi \circ \tilde{s}_{n,N} \circ \psi)(u^N_{jk} + \mathcal{I}) = u^N_{jk} + \mathcal{I}$$

for all j and k, so $\varphi \circ \tilde{s}_{n,N} \circ \psi = \mathrm{id}_{\mathcal{A}_N/\mathcal{I}}$. It follows that the algebra epimorphism ψ is injective and thus an isomorphism. Since $\mathcal{I} \subset \mathcal{K}$, this implies that $\mathcal{I} = \mathcal{K}$. □

We next establish relations between the values taken on generators, for a given cocycle on $S\mathcal{U}_q(N)$.

Lemma 2.28. *Let π be a representation of $S\mathcal{U}_q(N)$, and let η be a π-ε-cocycle. For $i < l \leqslant N$ and $j,k < N$,*

$$\eta(u_{il}) = -(I - q\pi(u_{ll}))^{-1}\pi(u_{il})\eta(u_{ll}), \tag{2.12a}$$

$$\eta(u_{li}) = -(I - q\pi(u_{ll}))^{-1}\pi(u_{li})\eta(u_{ll}), \tag{2.12b}$$

$$\begin{aligned}\pi(u_{NN} - 1)\eta(u_{jk}) = \Big(&\pi(u_{jk} - \delta_{jk}1) \\ &- (q^{-1} - q) \times \pi(1 - q^2 u_{NN})^{-1}\pi(u_{il}u_{li})\Big)\eta(u_{NN}).\end{aligned} \tag{2.12c}$$

In particular, η is determined by its value $\eta(u_{NN})$ when $\pi(1 - u_{NN})$ is injective, by Remark 2.26.

Proof. If $a = u_{il}$ or $a = u_{li}$ where $i < l \leqslant N$, then $a \in \ker\varepsilon$ and, by identities (2.6a) and (2.6b), $au_{ll} = qu_{ll}a$. Hence, by the cocycle property, $\pi(a)\eta(u_{ll}) + \eta(a) = q\pi(u_{ll})\eta(a)$. Since $\pi(u_{ll})$ is a contraction, this is equivalent to the identity $\eta(a) = -(I - q\pi(u_{ll}))^{-1}\pi(a)\eta(u_{ll})$.

By the cocycle property applied to identity (2.6d), if $j,k < N$ then

$$\begin{aligned}&\pi(u_{jk})\eta(u_{NN}) + \eta(u_{jk}) \\ &\qquad = \eta(u_{jk}u_{NN}) = \eta(u_{NN}u_{jk}) - (q^{-1} - q)\eta(u_{jN}u_{Nk}) \\ &\qquad = \pi(u_{NN})\eta(u_{jk}) + \eta(u_{NN})\varepsilon(u_{jk}) - (q^{-1} - q)\pi(u_{jN})\eta(u_{Nk}),\end{aligned}$$

so

$$
\begin{aligned}
&\pi(u_{NN}-1)\eta(u_{jk})\\
&= \pi(u_{jk}-\delta_{jk}1)\eta(u_{NN}) + (q^{-1}-q)\pi(u_{jN})\eta(u_{Nk})\\
&= (\pi(u_{jk}-\delta_{jk}1) - (q^{-1}-q)\pi(u_{jN})(I-q\pi(u_{NN}))^{-1}\pi(u_{Nk}))\eta(u_{NN})\\
&= (\pi(u_{jk}-\delta_{jk}1) - (q^{-1}-q)(I-q^2\pi(u_{NN}))^{-1}\pi(u_{jN}u_{Nk}))\eta(u_{NN}). \qquad \square
\end{aligned}
$$

We end this section by characterising those representations and cocycles on $\mathcal{SU}_q(N)$ that live on $\mathcal{SU}_q(n)$, for $n < N$.

Proposition 2.29. *Let π be a representation of $\mathcal{SU}_q(N)$, let η be a π-ε-cocycle, and let $n < N$.*

(a) *The following are equivalent:*
   (i) *π lives on $\mathcal{SU}_q(n)$.*
   (ii) *$\pi(u_{kj}) = \delta_{kj}I$ if $\max\{j,k\} > n$.*
   (iii) *$\pi(u_{jj}) = I$ for $n < j \le N$.*

(b) *Suppose π lives on $\mathcal{SU}_q(n)$. Then, the following are equivalent:*
   (i) *η lives on $\mathcal{SU}_q(n)$.*
   (ii) *$\eta(u_{kj}) = 0$ if $\max\{j,k\} > n$.*
   (iii) *$\eta(u_{jj}) = 0$ for $n < j \le N$.*

Proof. For both parts, the equivalence of (i) and (ii) follows from Proposition 2.27 because (ii) says π, respectively η, vanishes on the set $S_{n,N}$ (in the latter case, since cocycles kill the identity element); moreover, (ii) obviously implies (iii).

(a) For all $j = 1,\ldots,N$, the unitarity relations (2.7) imply the identities

$$
\begin{aligned}
\pi(u_{jj})^*\pi(u_{jj}) + \sum_{k\neq j}\pi(u_{kj})^*\pi(u_{kj}) &= I\\
&= \pi(u_{jj})\pi(u_{jj})^* + \sum_{k\neq j}\pi(u_{jk})\pi(u_{jk})^*,
\end{aligned}
$$

so if $\pi(u_{jj}) = I$, then $\pi(u_{kj}) = 0$ for $k \neq j$. Thus, (iii) implies (ii).

(b) By identities (2.12a) and (2.12b), if $\eta(u_{ll}) = 0$ then $\eta(u_{il}) = 0 = \eta(u_{li})$ for $i < l$, and so (iii) implies (ii). $\square$

3. Classification of Gaussian Generating Functionals

In this section, we investigate the Gaussian generating functionals on $\mathcal{SU}_q(N)$ and their Schürmann triples. We follow Procedure 2.6 for Gaussian representations, that is, representations of the form $\iota_{\mathsf{h}} \circ \varepsilon : a \mapsto \varepsilon(a)I_{\mathsf{h}}$. Since Gaussian cocycles vanish on K_2, we seek a Hermitian basis extension E from K_2 to K (see Section 2).

Lemma 3.1. *Set $v_j := (u_{jj}-1) \in K$ and*

$$
d_j := (2\mathrm{i})^{-1}(u_{jj}-u_{jj}^*) = (2\mathrm{i})^{-1}(v_j - v_j^*) \in K.
$$

Then, the following hold:

(a) $u_{jk} \in K_2$ *for* $j \neq k$,
(b) $v_j + v_j^* \in K_2$,
(c) $d_1 + \cdots + d_N \in K_2$,
(d) $d_j d_k - d_k d_j \in K_3$.

Proof. (a) Let $j \neq k$. Combining relations (2.6a) and (2.6b), one has $u_{jk}u_{ll} = qu_{ll}u_{jk}$ for $j \neq k$ and $l := \max(j,k)$. Therefore, since $u_{ll} - 1, u_{jk} \in K$,

$$u_{jk} = (1-q)^{-1}q(u_{ll}-1)u_{jk} - u_{jk}(u_{ll}-1) \in K_2.$$

(b) By the unitarity relation (2.7) we see that $1 - u_{jj}u_{jj}^* = \sum_{m \neq j} u_{jm}u_{jm}^* \in K_2$, so

$$v_j + v_j^* = (u_{jj}-1) + (u_{jj}-1)^* = -(1-u_{jj}u_{jj}^*) - (u_{jj}-1)(u_{jj}-1)^* \in K_2.$$

(c) Observe that

$$u_{11}\cdots u_{NN} = (v_1+1)\cdots(v_N+1) = 1 + (v_1 + \cdots + v_N) + \text{terms in } K_2.$$

Therefore, $v_1 + \cdots + v_N + (1 - u_{11}\cdots u_{NN}) \in K_2$. Since $D_q = 1$, we have

$$1 - u_{11}\cdots u_{NN} = \sum_{\sigma \in S_N,\, \sigma \neq \mathrm{id}} (-q)^{i(\sigma)} u_{1,\sigma(1)} \cdots u_{N,\sigma(N)}. \tag{3.1}$$

Now, for $\sigma \neq \mathrm{id}$ there is at least one j such that $j \neq \sigma(j)$, so, from part (a), the righthand side of (3.1) is in K_2. Thus, $v_1 + \cdots + v_N \in K_2$, hence,

$$d_1 + \cdots + d_N = (2\mathrm{i})^{-1}((v_1 + \cdots + v_N) - (v_1 + \cdots + v_N)^*) \in K_2.$$

(d) This follows from part (a), in view of the relations (2.6d) and (2.8a). ☐

Now, consider the family of characters determined by

$$\varepsilon_{\theta_2,\ldots,\theta_N}(u_{kl}) := e^{\mathrm{i}\theta_k}\delta_{k,l} \quad (k,l \in \{1,\ldots,N\}),$$

for $\theta_2,\ldots,\theta_N \in \mathbb{R}$ and θ_1 given implicitly by $\sum_{k=1}^N \theta_k = 0$. The pointwise defined linear functionals

$$\varepsilon_j' := \frac{\partial}{\partial\theta_j}\bigg|_{\theta_2=\cdots=\theta_N=0} \varepsilon_{\theta_2,\ldots,\theta_N} \quad (j = 2,\ldots,N) \tag{3.2}$$

are drifts because they kill 1 (since each $\varepsilon_{\theta_1,\ldots,\theta_d}$ is a character), vanish on K_2 (by Leibniz's rule, as $\varepsilon_{0,\ldots,0} = \varepsilon$) and are Hermitian (since $d_k^* = d_k$ and $\varepsilon_j'(d_k) = \delta_{jk}$).

Lemma 3.2. *Set* $E := \{d_2, \dots, d_N\}$. *Then, the following hold:*

(a) *E is a Hermitian basis extension from K_2 to K.*

(b) *$\{\varepsilon_j' : j = 2, \dots, N\}$ is a basis for the real space of drifts on $\mathcal{SU}_q(N)$.*

Proof. The set E is Hermitian, and it follows from parts (a), (b), and (c) of Lemma 3.1 that $E \cup K_2$ spans K. For $j, k = 2, \dots, N$, $\varepsilon_j'(d_k) = \delta_{jk}$, so E is linearly independent, and ε_j' kills K_2 so E and K_2 are disjoint. Thus, (a) holds, and so does (b) since drifts vanish on $\{1\} \cup K_2$. ◻

In view of part (d) of Lemma 3.1 and Corollaries 2.13 and 2.12, we deduce the following result.

Theorem 3.3. $\mathrm{SU}_q(N)$ *does not have property* (GC) *unless* $N \leqslant 2$.

This is also proved in [2]. $\mathrm{SU}_q(N)$ has property (AC) if $N = 2$ ([23], [21]).

From now on, we fix the Hermitian basis extension $E_N := \{d_2, \dots, d_N\}$ from K_2 to K, and thereby also the projection in $P_2(\mathcal{SU}_q(N))$ as in (2.4), which we denote P_N. The resulting family of projections is compatible with the subgroup relations $\mathrm{SU}_q(N) \geqslant \mathrm{SU}_q(n)$, as verified next.

Proposition 3.4. $P_n \circ s_{n,N} = s_{n,N} \circ P_N$ *for* $n < N$.

Proof. The epimorphism s_N (see (2.10)) sends d_N to 0 and, for $2 \leqslant n \leqslant N-1$, sends the d_n of $\mathcal{SU}_q(N)$ to the d_n of $\mathcal{SU}_q(N-1)$, so $s_N(E_N) = E_{N-1} \cup \{0\}$. Therefore, by Lemma 2.25, $P_{N-1} \circ s_N = s_N \circ P_N$. By identity (2.11) this can be iterated to yield the proposition. ◻

Note that the ε_j' obtained in (3.2) coincide with the functionals ε_d' ($d = d_j$) defined in (2.3) from the basis extension E_N. Thus, Proposition 2.10 yields the following characterization.

Proposition 3.5. *The Gaussian cocycles on $\mathcal{SU}_q(N)$ are precisely the maps of the form*

$$\eta = \sum_{j=2}^{N} \xi_j \varepsilon_j'(\cdot) \tag{3.3}$$

for a family of vectors $(\xi_j)_{j=2}^N$ in a Hilbert space h.

We next describe the Gaussian generating functionals on $\mathcal{SU}_q(N)$. Consider the pointwise-defined functionals

$$\varepsilon_{jk}'' := \frac{\partial^2}{\partial\theta_j\,\partial\theta_k}\bigg|_{\theta_2=\cdots=\theta_N=0} \varepsilon_{\theta_2,\dots,\theta_N} \quad (j, k = 2, \dots, N).$$

Theorem 3.6. *If we let $M_n(\mathbb{R})_+$ denote the set of real nonnegative-definite $n \times n$ matrices, the prescription*

$$(r, R) \mapsto \sum_{j=2}^{N} r_j \varepsilon_j' + \frac{1}{2} \sum_{j,k=2}^{N} r_{jk} \varepsilon_{jk}''$$

defines a bijection from $\mathbb{R}^{N-1} \times M_{N-1}(\mathbb{R})_+$ to the set of Gaussian generating functionals γ on $\mathcal{SU}_q(N)$ in which the second sum is the P_N-invariant component $\gamma \circ P_N$.

Proof. In view of Lemma 3.2, it suffices for us to verify that the prescription $[r_{jk}] \mapsto \frac{1}{2}\sum_{j,k=2}^{N} r_{jk}\varepsilon''_{jk}$ defines a bijection from $M_{N-1}(\mathbb{R})_+$ to the set of P_N-driftless (i.e., P_N-invariant) Gaussian generating functionals γ.

First note that by Leibniz's rule, for $a, b \in \mathcal{SU}_q(N)$,

$$\varepsilon''_{jk}(ab) = \varepsilon''_{jk}(a)\varepsilon(b) + \varepsilon'_j(a)\varepsilon'_k(b) + \varepsilon'_k(a)\varepsilon'_j(b) + \varepsilon(a)\varepsilon''_{jk}(b).$$

It follows that ε''_{jk} vanishes on K_3 and, by direct computation, $\varepsilon''_{jk}(d_l) = 0$ and $\varepsilon''_{jk}(d_l d_m) = \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}$ for $j, k, l, m = 2, \dots, N$. In particular, $\varepsilon''_{jk} \circ P_N = \varepsilon''_{jk}$ and, for all $c \in K$ and $\lambda \in \mathbb{C}^{N-1}$, $\sum \bar{\lambda}_j \varepsilon''_{jk}(c^*c)\lambda_k = 2|\sum \lambda_k \varepsilon'_k(c)|^2 \geqslant 0$ so, since nonnegative-definiteness is preserved under the Schur product, for any matrix $R = [r_{jk}] \in M_{N-1}(\mathbb{R})_+$ the functional $\frac{1}{2}\sum r_{jk}\varepsilon''_{jk}$ is conditionally positive and therefore a P_N-invariant Gaussian generating functional.

Conversely, if γ is a Gaussian generating functional, its associated cocycle η is of the form (3.3) and so, by Corollary 2.13 and part (d) of Lemma 3.1, η is Hermitian; hence, the Gram matrix $[\langle \xi_j, \xi_k \rangle]$ is real and thus in $M_{N-1}(\mathbb{R})_+$. ◻

Remarks 3.7. The CQG algebra $\mathcal{T}_{N-1}$ of the torus $\mathbb{T}^{N-1}$ is generated, as unital *-algebra, by a family of commuting unitaries $\{u_j : j = 1, \dots, N\}$ subject to the relation $u_1 \cdots u_N = 1$. The prescription $u_{jk} \mapsto \delta_{jk}u_j$ determines a CQG epimorphism $\tau_N : \mathcal{SU}_q(N) \to \mathcal{T}_{N-1}$ with respect to which the characters $\varepsilon_{\theta_2,\dots,\theta_N}$ of $\mathcal{SU}_q(N)$ live on $\mathbb{T}^{N-1}$. Therefore, the Gaussian generating functionals of $\mathcal{SU}_q(N)$ live on $\mathcal{T}_{N-1}$. It also follows that, for any compact quantum group $\mathbb{G}$ satisfying $\mathrm{SU}_q(N) \geqslant \mathbb{G} \geqslant \mathbb{T}^{N-1}$, the projection $P \in P_2(\mathcal{G})$ may be chosen to be compatible with those for $\mathcal{SU}_q(N)$ and $\mathcal{T}_{N-1}$, and the Gaussian generating functionals of $\mathcal{G}$ correspond to those of $\mathcal{T}_{N-1}$. Application of results on classical compact Lie groups in [24] to $\mathbb{T}^{N-1}$ gives an alternative proof of Theorem 3.6. The preliminary version of our paper ([9]) motivated generalisation of the theorem to all q-deformations of simply connected semisimple compact Lie groups (Theorem 6.1 in [7]).

4. DECOMPOSITION

This is the central section of the paper. We decompose an arbitrary representation π of $\mathcal{SU}_q(N)$ uniquely into a direct sum $\pi_1 \oplus \cdots \oplus \pi_N$, in which $\pi_1 = \pi_G$, as defined in Proposition 2.16 and, for $2 \leqslant n \leqslant N$, π_n lives on $\mathcal{SU}_q(n)$ and $\pi_n(1 - u_{nn})$ is injective. We then show that, in the corresponding decomposition $\eta_1 \oplus \cdots \oplus \eta_N$ of a π-ε-cocycle η, for $2 \leqslant n \leqslant N$ each cocycle η_n is approximately inner and determined by the vector $\eta(u_{nn})$. This implies that $\mathrm{SU}_q(N)$ has property (NAI) and so also (LK). We deduce a Hunt formula for $\mathrm{SU}_q(N)$ incorporating *full* decomposition for generating functionals.

The following elementary lemma plays a key role in the approximation of cocycles (part (a) is well known in, for example, ergodic theory). For bounded operators T, we write $\overline{\operatorname{ran}}\,T$ for $\overline{\operatorname{ran}\,T}$.

Lemma 4.1 (Contraction operator lemma). *For any contraction operator C on a Hilbert space, the following hold:*

(a) $\ker(I-C^*) = \ker(I-C)$, *so also* $\overline{\operatorname{ran}}(I-C) = \ker(I-C)^\perp = \overline{\operatorname{ran}}(I-C^*)$.

(b) *Setting* $P := P_{\ker(I-C)}$, *as* $t \to 1^-$, *we have* $P(t) := (1-t)(I-tC)^{-1} \xrightarrow{\text{SOT}} P$ *and* $P^\perp(t) := (I-tC)^{-1}(I-C) \xrightarrow{\text{SOT}} P^\perp$.

In particular, the following four conditions are equivalent:

(i) $I - C$ *is injective.*

(i)′ $I - C$ *has dense range.*

(ii) $(I-tC)^{-1}(I-C) \xrightarrow{\text{SOT}} I$ *as* $t \to 1^-$.

(ii)′ $(1-t)(I-tC)^{-1} \xrightarrow{\text{SOT}} 0$ *as* $t \to 1^-$.

Proof. (a) Let $\xi \in \ker(I-C) = \operatorname{ran}(I-C^*)^\perp = \operatorname{ran}(C^*-I)^\perp$. By symmetry, it suffices to prove that $\xi \in \ker(I-C^*)$. This follows by Pythagoras:

$$\|\xi\|^2 + \|(C^*-I)\xi\|^2 = \|C^*\xi\|^2 \leqslant \|\xi\|^2.$$

(b) For $0 < t < 1$, the following hold:

(1) $I - P(t) = tP^\perp(t)$.

(2) $\|P(t)\| \leqslant 1$.

(3) $P(t)(I-C) = (1-t)P^\perp(t)$.

Thus,

(4) $\|P(t)(I-C)\| \leqslant 2(1-t)/t$.

By (1), $P(t) \to I$ on $\ker P^\perp(t) = \ker(I-C)$ and, by (4) and (2), $P(t) \to 0$ on $\overline{\operatorname{ran}}(I-C)$. Hence, $P(t) \xrightarrow{\text{SOT}} P$ by (a), and so $P^\perp(t) \xrightarrow{\text{SOT}} P^\perp$ by (1). ◻

Decomposition of representations and cocycles. We start by separating out the maximal subspace on which the operator $\pi(1-u_{NN})$ acts injectively, for a given representation π.

Lemma 4.2. *Let π be a representation of $\mathcal{SU}_q(N)$. Then, π has a unique decomposition $\pi^N \oplus \pi_N$ for which π^N lives on $\mathcal{SU}_q(N-1)$ (which is equivalent to $\pi^N(1-u_{NN}) = 0$), and $\pi_N(1-u_{NN})$ is injective. Also,* $\mathsf{h}^{\pi^N} = \ker\pi(1-u_{NN})$.

Proof. The equivalence is contained in Proposition 2.29. We first show that $\mathsf{k} := \ker\pi(1-u_{NN})$ is an invariant subspace for π. Since the u_{jk} generate $\mathcal{SU}_q(N)$ as an algebra (Remark 2.26), to see this it suffices to fix $\xi \in \mathsf{k}$ and $j,k \in \{1,\dots,N\}$, and to verify that $\pi_{jk}\xi \in \mathsf{k}$ (in the convenient abbreviation $\pi_{jk} := \pi(u_{jk})$). For $j = k = N$ this is obvious. For $k < N$, applying π to identity (2.7), and then the vector functional ω_ξ, we see that $\pi_{Ns}^*\xi = 0 = \pi_{sN}\xi$ for $s < N$, so, by identity (2.8d),

$$\pi_{Nk}^*\pi_{Nk}\xi = \pi_{Nk}\pi_{Nk}^*\xi + (1-q^2)\sum_{m<k}\pi_{Nm}\pi_{Nm}^*\xi = 0.$$

Thus, $\pi_{Nk}\xi = 0$. Similarly, $\pi_{jN} = 0$. Finally, for $j, k < N$,

$$\pi_{jk}\pi_{NN}\xi = \pi_{NN}\pi_{jk}\xi - (q^{-1}-q)\pi_{jN}\pi_{Nk}\xi$$

by identity (2.6d), so $\pi_{jk}\pi_{NN}\xi = \pi_{NN}\pi_{jk}\xi$, in other words $\pi_{jk}\xi \in \mathsf{k}$, as required.

In the resulting decomposition, $\pi = \pi^N \oplus \pi_N$,

$$\pi^N(1-u_{NN}) = 0 \quad \text{and} \quad \pi_N(1-u_{NN}) \text{ is injective.}$$

It remains to prove uniqueness. Thus, let $\rho \oplus \sigma$ be another such decomposition of π; we must show that $\mathsf{h}^\rho = \mathsf{k}$. This follows from Lemma 4.1:

$$\begin{aligned}\mathsf{h}^\rho &= \ker \rho(1-u_{NN}) \subset \mathsf{k}\\ &= \operatorname{ran}\pi(1-u_{NN})^\perp \subset \operatorname{ran}\sigma(1-u_{NN})^\perp\\ &= (\mathsf{h}^\sigma)^\perp = \mathsf{h}^\rho.\end{aligned}$$ ◻

Definition 4.3. A decomposition $\pi_1 \oplus \cdots \oplus \pi_N$ of a representation of $S\mathcal{U}_q(N)$ is *full* if the following hold:

(1) For $1 \leqslant n < N$, there is a representation $\tilde{\pi}_n$ of $S\mathcal{U}_q(n)$ such that $\pi_n = \tilde{\pi}_n \circ s_{n,N}$.
(2) For $n \geqslant 2$, $\pi_n(1-u_{nn})$ is injective.

For $n = 1$, (1) says that π_n is Gaussian, and for $n \geqslant 2$, (1) implies that $\pi_n(1-u_{nn}) = \tilde{\pi}_n(1-u_{nn}^n)$ where u_{nn}^n denotes u_{nn} in $S\mathcal{U}_q(n)$. (2) is equivalent to $\pi(1-u_{nn})$ having dense range for $n \geqslant 2$.

This superscript convention, indicating which quantum subgroup is being referred to, continues below.

Theorem 4.4. *Every representation of $S\mathcal{U}_q(N)$ has a unique full decomposition.*

Proof. We prove this by induction on N. For $N = 1$ there is nothing to prove. Suppose the proposition holds for $N = K-1$ for some $K \geqslant 2$, and let π be a representation of $S\mathcal{U}_q(K)$.

Existence. By Lemma 4.2, $\pi = \pi^K \oplus \pi_K$ where $\pi_K(1-u_{KK})$ is injective and $\pi^K = \tilde{\pi} \circ s_K$ for a representation $\tilde{\pi}$ of $S\mathcal{U}_q(K-1)$. By the induction hypothesis, $\tilde{\pi} = \rho_1 \oplus \cdots \oplus \rho_{K-1}$ where ρ_1 is Gaussian and, for $k = 2, \dots, K-1$, $\rho_k(1-u_{kk}^{K-1})$ is injective and $\rho_k = \tilde{\rho}_k \circ s_{k,K-1}$, for some representation $\tilde{\rho}_k$ of $S\mathcal{U}_q(k)$. Set $\pi_k := \rho_k \circ s_K$ for $k = 1, \dots, K-1$. Then, $\pi = \pi_1 \oplus \cdots \oplus \pi_K$, where π_1 is

Gaussian, $\pi_K(1-u_{KK})$ is injective and, for $k=2,\dots,K-1$, $\pi_k(1-u_{kk})$ equals $\rho_k(1-u_{kk}^{K-1})$ and so is injective, and $\pi_k=\tilde{\rho}_k\circ s_{k,K-1}\circ s_K=\tilde{\rho}_k\circ s_{k,K}$, so π_k lives on $S\mathcal{U}_q(k)$.

Uniqueness. Suppose $\pi=\rho_1\oplus\cdots\oplus\rho_K$ is another such decomposition. Then, by the uniqueness part of Lemma 4.2, we have $\rho_K=\pi_K$ and $\rho_1\oplus\cdots\oplus\rho_{K-1}=\pi_1\oplus\cdots\oplus\pi_{K-1}$. Now, for $k=1,\dots,K-1$, $\pi_k=\tilde{\pi}_k\circ s_K$ and $\rho_k=\tilde{\rho}_k\circ s_K$ for representations $\tilde{\pi}_1,\dots,\tilde{\rho}_{K-1}$ of $S\mathcal{U}_q(K-1)$ and, by the surjectivity of s_K, $\tilde{\pi}_1\oplus\cdots\oplus\tilde{\pi}_{K-1}=\tilde{\rho}_1\oplus\cdots\oplus\tilde{\rho}_{K-1}$. Since $\tilde{\pi}_1$ and $\tilde{\rho}_1$ are Gaussian and, for $k=2,\dots,K-1$, $\tilde{\pi}_k$ and $\tilde{\rho}_k$ live on $S\mathcal{U}_q(k)$ and $\tilde{\pi}_k(1-u_{kk}^{K-1})$ and $\tilde{\rho}_k(1-u_{kk}^{K-1})$ are injective, it follows from the induction hypothesis that $\tilde{\pi}_k=\tilde{\rho}_k$ for $k=1,\dots,K-1$. Therefore, $\pi_k=\rho_k$ for $k=1,\dots,K$, as required. ◻

Theorem 4.5. *Let $\pi_1\oplus\cdots\oplus\pi_N$ be the full decomposition of a representation π of $S\mathcal{U}_q(N)$, and let $\eta_1\oplus\cdots\oplus\eta_N$ be the induced decomposition of a π-ε-cocycle η. Then, η_1 is Gaussian and, for $n\geqslant 2$, η_n lives on $S\mathcal{U}_q(n)$.*

Proof. For $n=1$, the cocycle η_n is Gaussian since the representation π_n is. For $m>n\geqslant 2$, by part (a) of Proposition 2.29 applied to identity (2.6d),

$$\begin{aligned}\pi_n(u_{nn})\eta_n(u_{mm})+\eta_n(u_{nn})&\\ &=\pi_n(u_{mm})\eta_n(u_{nn})+\eta_n(u_{mm})-(q^{-1}-q)\pi_n(u_{nm})\eta_n(u_{mn})\\ &=\eta_n(u_{nn})+\eta_n(u_{mm}),\end{aligned}$$

so $\eta_n(u_{mm})\in\ker\pi_n(1-u_{nn})=\{0\}$. Thus, by part (b) of Proposition 2.29, η_n lives on $S\mathcal{U}_q(n)$. ◻

Approximation of cocycles and (NAI) ***for*** $\mathrm{SU}_q(N)$**.** We now show that each of the cocycles η_n $(n\geqslant 2)$ in Theorem 4.5 is approximately inner.

Proposition 4.6. *Let η be a cocycle of a representation π of $S\mathcal{U}_q(N)$ such that $\pi(1-u_{NN})$ is injective. Then,*

$$\eta=\text{pw-lim}_{t\to 1^-}\eta_{\pi,\zeta(t)}\quad\text{where } \zeta(t):=-\pi(1-tu_{NN})^{-1}\eta(u_{NN}).$$

Proof. In view of the cocycle relations and Remark 2.26, it suffices to prove that, for each of the algebra generators $a=u_{jk}$, $\eta(a)$ is the pointwise limit as $t\to 1^-$ of the following expression:

$$-\pi(a-\varepsilon(a)1)\pi(1-tu_{NN})^{-1}\eta(u_{NN}). \tag{4.1}$$

We can prove this by using Lemma 4.1 (the contraction operator lemma) and Lemma 2.28.

Case $a=u_{NN}$. By Lemma 4.1, $\pi(1-tu_{NN})^{-1}\pi(1-u_{NN})\eta(u_{NN})\to\eta(u_{NN})$.

Case $a = u_{kN}$ or $a = u_{Nk}$ $(k < N)$. Then, $a \in \ker\varepsilon$ so $\pi(a) = \pi(a - \varepsilon(a)1)$. Thus, using relations (2.12a)–(2.12b), Lemma 4.1 implies that $\eta(a)$ equals

$$\begin{aligned}
&-\pi(1 - qu_{NN})^{-1}\pi(a)\eta(u_{NN}) \\
&\quad = -\lim_{t\to 1^-} \pi(1 - qu_{NN})^{-1}\pi(a)\pi(1 - tu_{NN})^{-1}\pi(1 - u_{NN})\eta(u_{NN}) \\
&\quad = -\lim_{t\to 1^-} \pi(a - \varepsilon(a))\pi(1 - tu_{NN})^{-1}\eta(u_{NN}).
\end{aligned}$$

Case $a = u_{jk}$ $(j, k < N)$. We must show that

$$-\pi(u_{jk} - \delta_{jk}1)\pi(1 - tu_{NN})^{-1}\eta(u_{NN}) \to \eta(u_{jk}).$$

By the contraction operator lemma,

$$-\pi(1 - tu_{NN})^{-1}\pi(u_{NN} - 1)\eta(u_{jk}) \to \eta(u_{jk}).$$

It therefore suffices to show that

$$-\pi(1-tu_{NN})^{-1}\pi(u_{NN}-1)\eta(u_{jk})+\pi(u_{jk}-\delta_{jk}1)\pi(1-tu_{NN})^{-1}\eta(u_{NN}) \to 0.$$

By identity (2.12c), the first term equals

$$-\pi(1-tu_{NN})^{-1}(\pi(u_{jk}-\delta_{jk}1)-(q^{-1}-q)\pi(1-q^2u_{NN})^{-1})\pi(u_{jN}u_{Nk})\eta(u_{NN}),$$

and so, since the operators $\pi(1 - q^2u_{NN})^{-1}$ and $\pi(1 - tu_{NN})^{-1}$ commute, after cancellation of the δ_{jk} terms and multiplication through by the invertible operator $\pi(1 - q^2u_{NN})$, we see that the task is equivalent to showing that the following family converges to 0, as $t \to 1^-$, on the vector $\eta(u_{NN})$:

$$\begin{aligned}
\text{(4.2)} \qquad &\pi(1 - q^2u_{NN})[\pi(u_{jk}), \pi(1 - tu_{NN})^{-1}] \\
&\qquad + (q^{-1} - q)\pi(1 - tu_{NN})^{-1}\pi(u_{jN}u_{Nk}).
\end{aligned}$$

We show that it converges to 0 strongly. Let us abbreviate $\pi(u_{il})$ to π_{il} for each i and l. It follows from identity (2.6d) that

$$[\pi_{jk}, \pi_{NN}^{\alpha}] = -(q^{-1} - q)\Big(\sum_{\nu=0}^{\alpha-1} q^{2\nu}\Big)\pi_{NN}^{\alpha-1}\pi_{jN}\pi_{Nk} \quad (\alpha \in \mathbb{Z}_+);$$

thus, taking the Neumann series for $(I - t\pi_{NN})^{-1}$, which is valid since $t\pi_{NN}$ is a strict contraction,

$$[\pi_{jk}, (I - t\pi_{NN})^{-1}] = -(q^{-1} - q)\sum_{\alpha=1}^{\infty}\sum_{\nu=0}^{\alpha-1} q^{2\nu}t^{\alpha}\pi_{NN}^{\alpha-1}\pi_{jN}\pi_{Nk}.$$

Therefore, (4.2) equals the following operator composed with $(q^{-1}-q)\pi_{jN}\pi_{Nk}$:

$$
\begin{aligned}
&-(I-q^2\pi_{NN})\sum_{\alpha=1}^{\infty}\sum_{\nu=0}^{\alpha-1} q^{2\nu}t^{\alpha}\pi_{NN}^{\alpha-1}+(I-t\pi_{NN})^{-1}\\
&\quad=\sum_{\nu=0}^{\infty}\sum_{\alpha=\nu+1}^{\infty}(q^{2(\nu+1)}(t\pi_{NN})^{\alpha}-tq^{2\nu}(t\pi_{NN})^{\alpha-1})+(I-t\pi_{NN})^{-1}\\
&\quad=\sum_{\nu=0}^{\infty}\Big((q^2t\pi_{NN})^{\nu+1}\sum_{\beta=0}^{\infty}(t\pi_{NN})^{\beta}-t(q^2t\pi_{NN})^{\nu}\sum_{\beta=0}^{\infty}(t\pi_{NN})^{\beta}\Big)\\
&\qquad\quad+(I-t\pi_{NN})^{-1}\\
&\quad=(I-q^2t\pi_{NN})^{-1}(q^2t\pi_{NN}-tI+I-q^2t\pi_{NN})(I-t\pi_{NN})^{-1}\\
&\quad=(I-q^2t\pi_{NN})^{-1}(1-t)(I-t\pi_{NN})^{-1},
\end{aligned}
$$

so the required convergence follows from Lemma 4.1. ◻

Theorem 4.7. *Let $\pi_1\oplus\cdots\oplus\pi_N$ be the full decomposition of a representation π of $SU_q(N)$, and let $\eta_1\oplus\cdots\oplus\eta_N$ be the induced decomposition of a π-ε-cocycle η. Then, for $n\geqslant 2$, $\eta_n=\text{pw-lim}_{t\to 1^-}\eta_{\pi_n,\xi(n,t)}$ where*

$$\xi(n,t):=-\pi_n(1-tu_{nn})^{-1}\eta_n(u_{nn}).$$

Thus, in terms of the decomposition $\mathsf{h}^{\pi}=\mathsf{h}^{\pi_G}\oplus\mathsf{h}^{\pi_R}$,

$$\eta=\text{pw-lim}_{t\to 1^-}\eta_G\oplus\eta_{\pi_R,\xi(t)}, \tag{4.3}$$

where

$$\xi(t):=-\pi_2(1-tu_{22})^{-1}\eta_2(u_{22})\oplus\cdots\oplus\pi_N(1-tu_{NN})^{-1}\eta_N(u_{NN}).$$

Proof. Let $n\geqslant 2$. By Theorem 4.5, $\eta_n=\tilde{\eta}_n\circ s_{n,N}$ for a cocycle $\tilde{\eta}_n$ on $SU_q(n)$ and, by Lemma 2.22, it suffices to prove that $\eta_{\tilde{\pi}_n,\xi(n,t)}$ converges pointwise to $\tilde{\eta}_n$. Now,

$$
\begin{aligned}
&\pi_n(1-u_{nn}) \quad \text{is injective (by Theorem 4.4)},\\
&\tilde{\pi}_n(1-tu_{nn}^n)=\pi_n(1-tu_{nn}) \quad \text{for all } t\in[0,1], \text{ and}\\
&\tilde{\eta}_n(u_{nn}^n)=\eta_n(u_{nn}),
\end{aligned}
$$

so $\tilde{\pi}_n(1-u_{nn}^n)$ is injective and $\xi(n,t)=-\tilde{\pi}_n(1-tu_{nn}^n)\tilde{\eta}_n(u_{nn}^n)$. The theorem therefore follows by applying Proposition 4.6 with $N=n$. ◻

Note that if π is completely non-Gaussian (and so $\mathsf{h}^{\pi_G}=\{0\}$), then (4.3) simplifies to the pointwise convergence $\eta_{\pi.\xi(t)}\to\eta$ as $t\to 1^-$, and so we draw the following immediate corollary.

Theorem 4.8. $\mathrm{SU}_q(N)$ *has property* (NAI), *and thus also* (LK).

Decomposition of generating functionals and Hunt formula for $\mathrm{SU}_q(N)$**.**

Lemma 4.9. *Let* (π', η') *and* (π'', η'') *be cyclic representation-cocycle pairs on* $\mathcal{SU}_q(N)$ *such that* (π', η') *lives on* $\mathcal{SU}_q(N-1)$ *and* $\pi''(1-u_{NN})$ *is injective. Then, the following hold:*

(a) *The cocycle* η' *vanishes on* $(1-u_{NN})K$.
(b) *The set* $\eta''((1-u_{NN})K) = \pi''(1-u_{NN})\eta''(K)$ *is dense in* $\mathsf{h}^{\pi''}$.
(c) *The cocycle* $\eta' \oplus \eta''$ *is cyclic.*

Proof. (a) This follows since $\pi'(1-u_{NN}) = 0$ because π' lives on $\mathcal{SU}_q(N-1)$ and $1-u_{NN} \in \ker s_N$.

(b) By Lemma 4.1, $\pi''(1-u_{NN})$ has dense range so this follows from the cyclicity of η''.

(c) The cyclicity of $\eta' \oplus \eta''$ follows from that of η' and η'' since, for $c_1, c_2 \in K$, by part (b) there is a sequence (d_p) in K such that $\eta''((1-u_{NN})d_p) \to \eta''(c_2 - c_1)$, and by part (a) $\eta'((1-u_{NN})d_p) = 0$ for all p so

$$(\eta' \oplus \eta'')(c_1 + (1-u_{NN})d_p) = \begin{pmatrix} \eta'(c_1) \\ \eta''(c_1) + \eta''((1-u_{NN})d_p) \end{pmatrix}$$
$$\to \begin{pmatrix} \eta'(c_1) \\ \eta''(c_2) \end{pmatrix} \quad \text{as } p \to \infty. \qquad \square$$

Definition 4.10. Let $N \geqslant 2$. We say that a completely non-Gaussian generating functional γ on $\mathcal{SU}_q(N)$ is *gf-irreducible* if the following holds: for any generating functional decomposition $\gamma = \gamma' + \gamma''$, if γ' lives on $\mathcal{SU}_q(N-1)$, then it is a drift.

Proposition 4.11. *Let* γ *be a generating functional on* $\mathcal{SU}_q(N)$ *for* $N \geqslant 2$, *and let* (π, η, γ) *be its Schürmann triple. Then,* γ *is gf-irreducible if and only if* $\pi(1-u_{NN})$ *is injective.*

Proof. Suppose first that γ is gf-irreducible. By Theorems 4.4 and 4.5 and Propositions 4.6 and 2.8, π and η decompose as $\pi^N \oplus \pi_N$ and $\eta^N \oplus \eta_N$, where $\pi_N(1-u_{NN})$ is injective, η^N lives on $\mathcal{SU}_q(N-1)$, and η_N is approximately inner and so completable by a P_N-invariant generating functional γ_N. The normalised Hermitian functional $\gamma^N := \gamma - \gamma_N$ satisfies $\gamma^N(c^*c) = \|\eta(c)\|^2 - \|\eta_N(c)\|^2 = \|\eta^N(c)\|^2$ for all $c \in K$, and so is a generating functional which completes η^N and thus also lives on $\mathcal{SU}_q(N-1)$, and satisfies $\gamma^N + \gamma_N = \gamma$. Thus, γ^N is a drift and so $\eta^N = 0$. But η^N is cyclic (since η is), and so $\mathsf{h}^{\pi^N} = \{0\}$; thus, $\pi = \pi_N$ and so $\pi(1-u_{NN})$ is injective.

Suppose conversely that $\pi(1-u_{NN})$ is injective, and let $\gamma' + \gamma''$ be a generating functional decomposition of γ such that γ' lives on $\mathcal{SU}_q(N-1)$. Let (π', η', γ') and $(\pi'', \eta'', \gamma'')$ be the Schürmann triples of γ' and γ''. Then, (π', η', γ') lives on $\mathcal{SU}_q(N-1)$, so η' vanishes on $(1-u_{NN})K$ by part (a) of

Lemma 4.9. Also, $(\pi' \oplus \pi'', \eta' \oplus \eta'', \gamma)$ is a Schürmann triple, so there is an isometry $V \in B(\mathsf{h}^{\pi}; \mathsf{h}^{\pi'} \oplus \mathsf{h}^{\pi''})$ such that $\binom{\eta'(c)}{\eta''(c)} = V\eta(c)$ for all $c \in K$. In view of part (b) of Lemma 4.9, these together imply that $\eta' = 0$, so γ' is a drift. Therefore, γ is gf-irreducible. ◻

Definition 4.12. A generating functional decomposition $\gamma = \gamma_1 + \cdots + \gamma_N$ on $\mathcal{SU}_q(N)$ is *full* if the following hold:

(1) For $1 \leqslant n < N$, $\gamma_n = \tilde{\gamma}_n \circ s_{n,N}$ for a generating functional $\tilde{\gamma}_n$ on $\mathcal{SU}_q(n)$.
(2) For $n \geqslant 2$, $\tilde{\gamma}_n$ is gf-irreducible and P_n-invariant.

For $n = 1$, (1) says that γ_n is Gaussian. Given (1), letting $(\tilde{\pi}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$ be $\tilde{\gamma}_n$'s Schürmann triple, so that

$$(\pi_n := \tilde{\pi}_n \circ s_{n,N}, \eta_n := \tilde{\eta}_n \circ s_{n,N}, \gamma_n := \tilde{\gamma}_n \circ s_{n,N})$$

is γ_n's Schürmann triple, the condition (2) is equivalent to

(2)′ For $n \geqslant 2$, $\pi_n(1 - u_{nn})$ is injective and γ_n is P_N-invariant.

This follows from Proposition 4.11 (since $\pi_n(1 - u_{nn}) = \tilde{\pi}_n(1 - u^n_{nn})$), and the compatibility of the projections P_n (Proposition 3.4).

Lemma 4.13. *If a generating functional* γ *on* $\mathcal{SU}_q(N)$ *has a full decomposition* $\gamma_1 + \cdots + \gamma_N$ *then, in terms of each* γ_n*'s Schürmann triple* $(\pi_n, \eta_n, \gamma_n)$*, the following hold:*

(a) $\pi_1 \oplus \cdots \oplus \pi_N$ *is a full (representation) decomposition.*
(b) *The cocycle* $\eta_1 \oplus \cdots \oplus \eta_N$ *is cyclic.*

Proof. Let $\gamma = \gamma_1 + \cdots + \gamma_N$ be such a decomposition. For each n denote by $(\tilde{\pi}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$ the induced Schürmann triple on $\mathcal{SU}_q(n)$, noting that for $n = 2, \ldots, N$, $\tilde{\gamma}_n$ is gf-irreducible and, by (2)′, the operator $\pi_n(1 - u_{nn})$ is injective and γ_n is P_N-invariant. In particular, (a) holds.

(b) For $N = 1$ there is nothing to prove. Suppose the proposition holds for $N = K - 1$ where $K \geqslant 2$, and that a generating functional γ on $\mathcal{SU}_q(K)$ has a full decomposition $\gamma = \gamma_1 + \cdots + \gamma_K$. In the above tilde notation, note that for $k = 1, \ldots, K - 1$,

$$(\hat{\pi}_k := \tilde{\pi}_k \circ s_{k,K-1}, \hat{\eta}_k := \tilde{\eta}_k \circ s_{k,K-1}, \hat{\gamma}_k := \tilde{\gamma}_k \circ s_{k,K-1})$$

is a cyclic Schürmann triple (since $(\pi_k, \eta_k, \gamma_k)$ is), and set $\hat{\gamma}^K := \hat{\gamma}_1 + \cdots + \hat{\gamma}_{K-1}$. This generating functional decomposition is full because $\hat{\gamma}_k = \tilde{\gamma}_k \circ s_{k,K-1}$ for each k and, for $k = 2, \ldots, K-1$, $\tilde{\gamma}_k$ is gf-irreducible and P_k-invariant. Therefore, by the induction hypothesis, $\hat{\eta}_1 + \cdots + \hat{\eta}_{K-1}$ is cyclic, which means that $\eta_1 \oplus \cdots \oplus \eta_{K-1}$ is cyclic; and so, by part (c) of Lemma 4.9,

$$\eta_1 \oplus \cdots \oplus \eta_K = (\eta_1 \oplus \cdots \oplus \eta_{K-1}) \oplus \eta_K$$

is too. Hence, (b) follows by induction. ◻

Theorem 4.14. *Every generating functional γ on $\mathcal{SU}_q(N)$ has a unique full decomposition.*

Proof. Existence. Let γ be a generating functional on $\mathcal{SU}_q(N)$, and (π, η, γ) be its Schürmann triple. By Theorem 4.4, π then has a full decomposition $\pi_1 \oplus \cdots \oplus \pi_N$; let $\eta_1 \oplus \cdots \oplus \eta_N$ be the corresponding decomposition of η. By Theorems 4.5 and 4.7, η_n lives on $\mathcal{SU}_q(n)$ for each n and, for $n = 2, \ldots, N$, η_n is approximately inner and thus completable by a P_N-invariant generating functional γ_n, so γ_n also lives on $\mathcal{SU}_q(n)$. Moreover, letting $(\tilde{\pi}, \tilde{\eta}, \tilde{\gamma})$ be the induced Schürmann triple on $\mathcal{SU}_q(n)$, $\tilde{\pi}_n(1 - u_{nn}^n)$ equals $\pi_n(1 - u_{nn})$ and so is injective; thus, $\tilde{\gamma}_n$ is gf-irreducible by Proposition 4.11. Now, the functional $\gamma_1 := \gamma - (\gamma_2 + \cdots + \gamma_N)$ is Hermitian and normalised, and satisfies $\gamma_1(c^*c) = \|\eta(c)\|^2 - (\|\eta_2(c)\|^2 + \cdots + \|\eta_N(c)\|^2) = \|\eta_1(c)\|^2$ for all $c \in K$, and so is a generating functional which completes η_1; moreover, it is Gaussian because π_1 is. It follows that $\gamma_1 + \cdots + \gamma_N$ is a full decomposition of γ.

Uniqueness. Let $\gamma_1 + \cdots + \gamma_N$ and $\gamma_1' + \cdots + \gamma_N'$ be full decompositions of a generating functional γ on $\mathcal{SU}_q(N)$. Set $\pi := \pi_1 \oplus \cdots \oplus \pi_N$ and $\eta := \eta_1 \oplus \cdots \oplus \eta_N$ where, for each n, $(\pi_n, \eta_n, \gamma_n)$ is γ_n's Schürmann triple—and do likewise for $\gamma_1', \ldots, \gamma_N'$. Since $\gamma_1 = \gamma - (\gamma_2 + \cdots + \gamma_N)$ and for $n \geqslant 2$, $\gamma_n \circ P_N = \gamma_n$ and $\gamma_n(c^*c) = \|\eta_n(c)\|^2$ for $c \in K$, and likewise for $\gamma_1', \ldots, \gamma_N'$, uniqueness follows once it is verified that $\|\eta_n(\cdot)\| = \|\eta_n'(\cdot)\|$ for $n \geqslant 2$. By Lemma 4.13, $\pi := \pi_1 \oplus \cdots \oplus \pi_N$ and $\pi' := \pi_1' \oplus \cdots \oplus \pi_N'$ are full (representation) decompositions and (π, η, γ) and (π', η', γ) are cyclic Schürmann triples. Therefore, there is a unitary operator $U \in B(\mathsf{h}^{\pi}; \mathsf{h}^{\pi'})$ such that $\eta' = U\eta(\cdot)$ and $\pi' = U\pi(\cdot)U^*$. The full decomposition $\pi = \pi_1 \oplus \cdots \oplus \pi_N$ evidently induces a full decomposition, say $\pi_1^U \oplus \cdots \oplus \pi_N^U$, of π'; the resulting decomposition $\eta' = \eta_1^U \oplus \cdots \oplus \eta_N^U$ satisfies $\|\eta_n^U(\cdot)\| = \|\eta_n(\cdot)\|$ for each n. Thus, by the uniqueness part of Theorem 4.4, for each n, $\pi_n^U = \pi_n'$ so $\eta_n^U = \eta_n'$; therefore, $\|\eta_n'(\cdot)\| = \|\eta_n(\cdot)\|$, as required. ◻

Combining the theorems of this section with Theorem 3.6 and Remarks 2.18 and 2.4, we deduce our main result.

Theorem 4.15 (Hunt formula for $\mathrm{SU}_q(N)$***).*** *Let γ be a generating functional on $\mathcal{SU}_q(N)$. Then, there is a unique decomposition $\gamma = \gamma_D + \gamma_G + \gamma_{NG}$, in which γ_D is a drift, and γ_G and γ_{NG} are P_N-invariant generating functionals which are, respectively, Gaussian and completely non-Gaussian. Moreover, the following hold:*

(1) *γ_G and γ_D are uniquely parameterised by a matrix in $M_{N-1}(\mathbb{R})_+$ and vector in $\mathbb{R}^{N-1}$.*

(2) *γ has a unique full decomposition $\gamma_1 + \cdots + \gamma_N$, and if $(\pi_n, \eta_n, \gamma_n)$ is γ_n's Schürmann triple for each n, then*

$$(\pi := \pi_1 \oplus \cdots \oplus \pi_N, \eta := \eta_1 \oplus \cdots \oplus \eta_N, \gamma)$$

is γ's Schürmann triple.

(3) $\gamma_{NG} = \text{pw-lim}_{t\to 1^-}\, \omega_{\xi(t)} \circ \pi_R \circ P_N$ *where* π_R *is the non-Gaussian remainder of* π *and*

$$\xi(t) := -\pi_2(1 - tu_{22})^{-1}\eta_2(u_{22}) \oplus \cdots \oplus \pi_N(1 - tu_{NN})^{-1}\eta_N(u_{NN}).$$

The realisation of γ_{NG} in (3) is analogous to that of γ_L in the classical Hunt formula (1.1) given in the remark following Proposition 2.8.

For the limiting case $q = 1$ corresponding to the compact Lie group $\mathrm{SU}(N)$ the proofs of Lemma 3.1 and Theorem 4.7, on which our Hunt formula depends, are no longer valid. However, the theorem as stated still holds. Indeed, the Gaussian/non-Gaussian decomposition and parameterisations (1) are statements of Hunt's results in the language of generating functionals; moreover, (2) is seen by decomposing the Lévy measure into its restrictions to the corresponding subgroups of $\mathrm{SU}(N)$.

5. From Parametrization by h^π to Quasi-innerness

Given a gf-irreducible generating functional on $\mathcal{SU}_q(N)$, with Schürmann triple (π, η, γ), by Proposition 4.11 and Lemma 2.28 we know that $\pi(1 - u_{NN})$ is injective and so η is determined by its value $\eta(u_{NN})$. One may therefore ask which vectors of the representation space h^π arise in this way. In case $N = 2$ every vector does, so the cocyles are parameterised by h^π ([23, Theorem 2.8], [21, Theorem 3.3]). We now show this to be false for $N = 3$; the argument extends to higher values of N. The section ends with an indication of a positive counterpart to this, namely, a quasi-innerness property of completely non-Gaussian cocycles/π-ε-derivations.

Proposition 5.1. *There is a representation* π *of* $\mathcal{SU}_q(3)$ *and vector* ξ *in* h^π *such that* $\pi(1 - u_{33})$ *is injective but* $\eta(u_{33}) \neq \xi$ *for every* π-ε-*cocycle* η.

Proof. Following Woronowicz, we write the generators u_{jk} of $\mathcal{SU}_q(2)$ as

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}.$$

Let ρ be the irreducible representation of $\mathcal{SU}_q(2)$ on $\ell^2(\mathbb{Z}_+)$ defined, in terms of the standard orthonormal basis $(e_n)_{n\geqslant 0}$, by

$$\rho(\alpha) : e_n \mapsto \sqrt{1 - q^{2n}}\, e_{n-1} \quad \text{and} \quad \rho(\gamma) : e_n \mapsto q^n e_n$$

(where $e_{-1} := 0$). For $k = 1, 2$, set $\rho_k := \rho \circ r_k$ for the CQG epimorphisms $r_k : \mathcal{SU}_q(3) \to \mathcal{SU}_q(2)$ given by

$$r_1 : [u_{jk}] \mapsto \begin{bmatrix} \alpha & -q\gamma^* & 0 \\ \gamma & \alpha^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad r_2 : [u_{jk}] \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & -q\gamma^* \\ 0 & \gamma & \alpha^* \end{bmatrix}$$

(so $r_1 = s_3$). Then, ρ_1 and ρ_2 are representations of $S\mathcal{U}_q(3)$ and so, setting $\pi := \rho_1 \star \rho_2$,

$$\begin{aligned}\left[\pi(u_{jk})\right]_{j,k} &= \left[\sum_{i=1}^{3} \rho_1(u_{ji}) \otimes \rho_2(u_{ik})\right]_{j,k} \\ &= \begin{bmatrix} \rho(\alpha)\otimes I & -q\rho(\gamma)^*\otimes\rho(\alpha) & q^2\rho(\gamma)^*\otimes\rho(\gamma)^* \\ \rho(\gamma)\otimes I & \rho(\alpha)^*\otimes\rho(\alpha) & -q\rho(\alpha)^*\otimes\rho(\gamma)^* \\ 0 & I\otimes\rho(\gamma) & I\otimes\rho(\alpha)^* \end{bmatrix}.\end{aligned}$$

Now, $\pi(1-u_{33}) = I \otimes \rho(1-\alpha^*)$ is injective because $\rho(1-\alpha^*)$ is. Suppose for a contradiction there is a π-ε-cocycle η satisfying $\eta(u_{33}) = e_0 \otimes e_0$. Since $\pi(u_{31}) = 0$ and $\rho(\alpha)e_0 = 0$, relation (2.12c) for $j = 1 = k$ implies

$$\begin{aligned}(I\otimes\rho(1-\alpha^*))\eta(u_{11}) &= (I-\pi(u_{33}))\eta(u_{11}) \\ &= (I-\pi(u_{11}))\eta(u_{33}) \\ &= \rho(1-\alpha)e_0\otimes e_0 = e_0\otimes e_0.\end{aligned}$$

For $n \geqslant 0$, set $a_n := \langle e_0 \otimes e_n, \eta(u_{11})\rangle$. Now $\rho(\alpha^*)e_n = \sqrt{1-q^{2(n+1)}}e_{n+1}$, so

$$\begin{aligned}e_0 &= \sum_{n\geqslant 0} a_n(I-\rho(\alpha^*)e_n \\ &= \sum_{n\geqslant 0} a_n\Big(e_n - \sqrt{1-q^{2(n+1)}}e_{n+1}\Big) \\ &= a_0e_0 + \sum_{n\geqslant 1}\Big(a_n - a_{n-1}\sqrt{1-q^{2n}}\Big)e_n.\end{aligned}$$

Thus, $a_0 = 1$ and, for $n \geqslant 1$, $|a_n|^2 = \prod_{k=1}^{n}(1-q^{2k})$. Therefore, since $\sum|a_n|^2 \leqslant \|\eta(u_{11})\|^2 < \infty$, $\prod_{k=1}^{n}(1-q^{2k}) \to 0$ as $n \to \infty$, so $\sum q^{2k}$ diverges and we have our contradiction. ◻

This leaves us with the question: which vectors in h^π may occur as values $\eta(u_{NN})$ for a cocycle η? Every element in the dense subspace $\operatorname{ran}\pi(1-u_{NN})$ occurs; and the collection of cocycles determined by them is precisely the set of coboundaries. Indeed, for $\xi' = -\pi(1-u_{NN})\xi$, by the contraction operator lemma we have (see Theorem 4.7) the following pointwise convergence as $t \to 1^-$:

$$-\pi\circ(\mathrm{id}-\iota\circ\varepsilon)(\cdot)\pi(1-tu_{NN})^{-1}\xi' \to \pi\circ(\mathrm{id}-\iota\circ\varepsilon)(\cdot)\xi = \eta_{\pi,\xi},$$

and the identity $\eta_{\pi,\xi}(u_{NN}) = \pi(u_{NN}-1)\xi = \xi'$.

Proposition 5.2. *Let $(\xi(\lambda))$ be a net in h^π. Then, the net of coboundaries $(\eta_{\pi,\xi(\lambda)})$ converges pointwise on $S\mathcal{U}_q(N)$ provided that it converges on u_{jj} for $1 \leqslant j \leqslant N$.*

Proof. This follows from Remark 2.26 since, setting $l := \max(j,k)$ for $j \neq k$, relations (2.6a) or (2.6b) imply that

$$\begin{aligned}\eta_{\pi,\xi(\lambda)}(u_{jk}) &= \pi(u_{jk})\xi(\lambda)\\ &= \pi(1-qu_{ll})^{-1}\pi(1-qu_{ll})\pi(u_{jk})\xi(\lambda)\\ &= -\pi(1-qu_{ll})^{-1}\pi(u_{jk})\pi(u_{ll}-1)\xi(\lambda)\\ &= -\pi(1-qu_{ll})^{-1}\pi(u_{jk})\eta_{\pi,\xi(\lambda)}(u_{ll}).\end{aligned}$$ ◻

We conclude this section with a quasi-innerness property enjoyed by all completely non-Gaussian cocycles.

Theorem 5.3. *Let π be a completely non-Gaussian representation of $S\mathcal{U}_q(N)$, and let $(\overline{\mathsf{h}^\pi}, J)$ denote the completion of h^π with respect to the norm*

$$|||\cdot|||:\xi \mapsto \Big(\sum_{j=1}^{N}\|\pi(1-u_{jj})\xi\|^2\Big)^{1/2}.$$

Then, a net $(\xi(\lambda))$ in h^π is $|||\cdot|||$-Cauchy if and only if the corresponding net of π-ε-coboundaries $(\eta_{\pi,\xi(\lambda)})$ converges pointwise. Moreover, the following hold:

(1) *There is a unique operator $\bar{\pi}: K \to B(\overline{\mathsf{h}^\pi};\mathsf{h}^\pi)$ which "extends" the representation π in the sense that it satisfies*

$$\bar{\pi}(ac) = \pi(a)\bar{\pi}(c) \text{ and } \bar{\pi}(c)J = \pi(c) \quad (a \in S\mathcal{U}_q(N),\ c \in K).$$

(2) *The prescription $\chi \mapsto \eta_{\bar{\pi},\chi} := (a \mapsto \bar{\pi}(a-\varepsilon(a)1)\chi)$ defines a linear isomorphism from $\overline{\mathsf{h}^\pi}$ to the space of π-ε-cocycles.*

There is also a unique operator $\bar{\bar{\pi}}: K_2 \to B(\overline{\mathsf{h}^\pi})$ such that

$$\bar{\bar{\pi}}(c^*c) = \bar{\pi}(c)^*\bar{\pi}(c) \text{ and } J^*\bar{\bar{\pi}}(e)J = \pi(e) \quad (c \in K,\ e \in K_2).$$

This has the property that, for all $\chi \in \overline{\mathsf{h}^\pi}$, the generating functional $\omega_\chi \circ \bar{\bar{\pi}} \circ P_N$ completes $(\pi, \eta_{\bar{\pi},\chi})$.

6. The Case of $\mathrm{U}_q(N)$

A Hunt formula for $\mathrm{U}_q(N)$ may be obtained by employing very similar arguments to those used above for $\mathrm{SU}_q(N)$. The upshot is the same as Theorem 4.15 except that it is with respect to the tower of subgroups $\mathrm{U}_q(0) \leqslant \cdots \leqslant \mathrm{U}_q(N)$ with $\mathrm{U}_q(0)$ denoting the trivial compact quantum group, rather than the tower $\mathrm{SU}_q(1) \leqslant \cdots \leqslant \mathrm{SU}_q(N)$ (also starting at the trivial group); thus, N replaces $N-1$ in (1), the decomposition in (2) starts at $n = 0$ rather than $n = 1$, and the components of $\xi(t)$ in (3) start at $n = 1$ rather than $n = 2$. We therefore instead discuss only the (NAI) and (GC) questions for $\mathrm{U}_q(N)$, as these may easily be deduced from our results and reasoning for the $\mathrm{SU}_q(N)$ quantum groups.

Since $\mathrm{SU}_q(N+1) \geqslant \mathrm{U}_q(N) \geqslant \mathbb{T}^N$, it follows from Remarks 3.7 that $\mathcal{U}_q(N)$ has the same Gaussian generating functionals as $\mathcal{SU}_q(N+1)$ and a Hermitian projection P' for $\mathcal{U}_q(N)$ compatible with that of $\mathcal{SU}_q(N+1)$ is the one corresponding to the following choice of basis extension:

$$E' = \{t_N(d_n) : 2 \leqslant n \leqslant N\} \cup \{t_N(d_{N+1}) = (2\mathrm{i})^{-1}(D^{-1} - D^{-1*})\}.$$

Theorem 6.1. $\mathrm{U}_q(N)$ *does not have property* (GC), *unless* $N = 1$.

Proof. The reasoning used in the proof of the $\mathcal{SU}_q(N)$ counterpart (Theorem 3.3) applies. By part (d) of Lemma 3.1, the basis extension E' again consists of elements whose commutators lie in K_3, and $\dim K/K_2 = N \geqslant 2$ unless $N = 1$ so Corollaries 2.13 and 2.12 again apply. ◻

Since the (NAI) property is hereditary (Proposition 2.23) and $\mathrm{SU}_q(N+1)$ has it (Theorem 4.8), $\mathrm{U}_q(N)$ does too.

Theorem 6.2. $\mathrm{U}_q(N)$ *has property* (NAI), *and thus also* (LK).

Acknowledgements. We gratefully acknowledge the MFO for a two-week Research-in-Pairs stay at Oberwolfach, where this work began. The first author was supported by the French "Investissements d'Avenir" programme (project ISITE-BFC, contract ANR-15-IDEX-03) and by an ANR project (no. ANR-19-CE40-0002). The second author was supported by the Polish National Science Center (grant SONATA 2016/21/D/ST1/03010) and by the Polish National Agency for Academic Exchange (under the framework of POLONIUM programme PPN/BIL/2018/1/00197/U/00021). The first and fourth authors were supported by the UK-France Alliance Hubert Curien Programme of research collaboration.

REFERENCES

[1] D. APPLEBAUM, *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, vol. 93, Cambridge University Press, Cambridge, 2004. https://dx.doi.org/10.1017/CBO9780511755323. MR2072890

[2] B. DAS, U. FRANZ, A. KULA, and A. SKALSKI, *Lévy-Khintchine decompositions for generating functionals on algebras associated to universal compact quantum groups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **21** (2018), no. 3, 1850017, pp. 36. https://dx.doi.org/10.1142/S0219025718500170. MR3860250

[3] S. DĂSCĂLESCU, C. NĂSTĂSESCU, and Ş. RAIANU, *Hopf Algebras: An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 235, Marcel Dekker, Inc., New York, 2001. MR1786197

[4] M. S. DIJKHUIZEN and T. H. KOORNWINDER, *CQG algebras: A direct algebraic approach to compact quantum groups*, Lett. Math. Phys. **32** (1994), no. 4, 315–330. https://dx.doi.org/10.1007/BF00761142. MR1310296

[5] U. FRANZ, *Lévy processes on quantum groups and dual groups*, Quantum Independent Increment Processes. II, Lecture Notes in Math., vol. 1866, Springer, Berlin, 2006, pp. 161–257. https://dx.doi.org/10.1007/11376637_3. MR2213449

[6] ______, *Independence and Lévy processes in quantum probability*, Noncommutative Mathematics for Quantum Systems, Cambridge-IISc Series, Cambridge University Press, Delhi, 2016. https://dx.doi.org/10.1017/CBO9781316562857.

[7] U. Franz, A. Freslon, and A. Skalski, *Connectedness and Gaussian parts for compact quantum groups*, J. Geom. Phys. **184** (2023), Paper No. 104710, pp. 18. https://dx.doi.org/10.1016/j.geomphys.2022.104710. MR4515816

[8] U. Franz, M. Gerhold, and A. Thom, *On the Lévy-Khinchin decomposition of generating functionals*, Commun. Stoch. Anal. **9** (2015), no. 4, 529–544. https://dx.doi.org/10.31390/cosa.9.4.06. MR3613811

[9] U. Franz, A Kula, J. M. Lindsay, and M. Skeide, *Hunt's formula for* $\mathrm{SU}_q(N)$ *and* $\mathrm{U}_q(N)$ (2020), preprint. https://dx.doi.org/10.48550/arXiv.2009.06323.

[10] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. II*: *Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups.*, Die Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 152, Springer-Verlag, New York-Berlin, 1970. MR0262773

[11] G. A. Hunt, *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264–293. https://dx.doi.org/10.2307/1992917. MR0079232

[12] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997. https://dx.doi.org/10.1007/978-3-642-60896-4. MR1492989

[13] H. T. Koelink, *On $*$-representations of the Hopf $*$-algebra associated with the quantum group* $\mathrm{U}_q(n)$, Compositio Math. **77** (1991), no. 2, 199–231. MR1091898

[14] J. M. Lindsay and A. G. Skalski, *Quantum stochastic convolution cocycles. I*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 3, 581–604 (English, with English and French summaries). https://dx.doi.org/10.1016/j.anihpb.2004.10.002. MR2139034

[15] P.-A. Meyer, *Quantum Probability for Probabilists*, Lecture Notes in Mathematics, vol. 1538, Springer-Verlag, Berlin, 1993. https://dx.doi.org/10.1007/978-3-662-21558-6. MR1222649

[16] S. Neshveyev and L. Tuset, *Compact Quantum Groups and their Representation Categories*, Cours Spécialisés [Specialized Courses], vol. 20, Société Mathématique de France, Paris, 2013. MR3204665

[17] M. Rosso, *Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif*, Duke Math. J. **61** (1990), no. 1, 11–40 (French). https://dx.doi.org/10.1215/S0012-7094-90-06102-2. MR1068378

[18] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original; Revised by the author. MR1739520

[19] M. Schürmann, *Gaussian states on bialgebras*, Quantum Probability and Applications, V (Heidelberg, 1988) (L. Accardi and W. von Waldenfels, eds.), Lecture Notes in Math., vol. 1442, Springer, Berlin, 1990, pp. 347–367. https://dx.doi.org/10.1007/BFb0085528. MR1091320

[20] ______, *White Noise on Bialgebras*, Lecture Notes in Mathematics, vol. 1544, Springer-Verlag, Berlin, 1993. https://dx.doi.org/10.1007/BFb0089237. MR1238942

[21] M. Schürmann and M. Skeide, *Infinitesimal generators on the quantum group* $\mathrm{SU}_q(2)$, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), no. 4, 573–598. https://dx.doi.org/10.1142/S0219025798000314. MR1665276

[22] M. Schürmann, M. Skeide, and S. Volkwardt, *Transformation of quantum Lévy processes on Hopf algebras*, Commun. Stoch. Anal. **4** (2010), no. 4, 553–577. https://dx.doi.org/10.31390/cosa.4.4.07. MR2743670

[23] M. Skeide, *The Lévy-Khintchine formula for the quantum group* $\mathrm{SU}_q(2)$, PhD thesis, Heidelberg, 1994, http://web.unimol.it/skeide/.

[24] ______, *Hunt's formula for* $\mathrm{SU}_q(2)$*—a unified view*, Open Syst. Inf. Dyn. **6** (1999), no. 1, 1–27. https://dx.doi.org/10.1023/A:1009644904638. MR1805268

[25] T. TIMMERMANN, *An Invitation to Quantum Groups and Duality*: *From Hopf Algebras to Multiplicative Unitaries and Beyond*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008. https://dx.doi.org/10.4171/043. MR2397671

[26] S. L. WORONOWICZ, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), no. 4, 613–665. MR0901157

[27] ______, *Twisted* $\mathrm{SU}(2)$ *group*: *An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117–181. https://dx.doi.org/10.2977/prims/1195176848. MR0890482

[28] ______, *Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted* $\mathrm{SU}(N)$ *groups*, Invent. Math. **93** (1988), no. 1, 35–76. https://dx.doi.org/10.1007/BF01393687. MR0943923

[29] ______, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR1616348

UWE FRANZ:
Université de Franche-Comté
CNRS, LmB, Besançon
France
E-MAIL: uwe.franz@univ-fcomte.fr

ANNA KULA:
Institute of Mathematics
University of Wroclaw
pl. Grunwaldzki 2/4
50-384 Wrocaw
Poland
E-MAIL: anna.kula@math.uni.wroc.pl

J.MARTIN LINDSAY:
Department of Mathematics and Statistics
Lancaster University
Fylde Ave
Bailrigg, Lancaster LA1 4YR
United Kingdom
E-MAIL: j.m.lindsay@lancaster.ac.uk

MICHAEL SKEIDE:
Dipartimento di Economia
Università degli Studi del Molise
Via De Sanctis
86100 Campobasso
Italy
E-MAIL: skeide@unimol.it

KEY WORDS AND PHRASES: Quantum Lévy process, convolution semigroup, compact quantum group, CQG algebra, generating functional, Schürmann triple.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 46L53, 60G51.

Received: June 28, 2021.