

Examiner report on *“Analysis and simulations of the non-linear models with non-local dispersal of species considered in finite, flat habitats”*, the PhD dissertation by Maciej Tadej

Patterns and scenarios of species spread (in the thesis referred to as dispersal) have been a major focus of research in theoretical and mathematical ecology for several decades. Revealing the role of dispersal is the core issue in many fundamental ecological problems. One such problem is pattern formation, in particular vegetation patterns. A timely and important question is how the pattern may be affected by the climate change and how that in turn may affect species persistence. Another important problem where dispersal is a central element arises in the context of biological invasion and is known as the problem of critical initial aggregation: how large must be the population size of the alien (introduced) species and/or how large must be the initially occupied domain so that the alien species would start spreading into new areas? Although significant progress has been done in understanding the above problems over the last few decades, due to the complexity of ecosystem interactions, many questions remain. One open question is the effects of nonlocality, and this is where the PhD thesis by Maciej Tadej intends to make an important contribution.

Mathematical modelling is commonly used as an efficient research approach to make an insight into the above problems. In particular, models consisting of partial differential equations have been widely used. However, such models almost entirely overlook the situation when different processes contributing to the dynamics occur on different spatial scales. Meanwhile, in case of vegetation patterns, which is the main ecological context for the thesis, the existence of at least two very different spatial scales (one for growth, the other for dispersal) is an inherent property of the system. The existence of multiple scales is best captured by nonlocal kernel-based model, in particular models described by integral-differential equations. This justifies the relevance of the approach used by Maciej.

The thesis is well structured and in general well written. In Chapter 1, the author provides an instructive and rather comprehensive introduction into the problem. Chapter 2 provides an instructive introduction to the spectral theory of integral operators at the level appropriate for the research presented in the thesis. In Chapter 3, a detailed investigation of the stationary problem is provided, with a special attention to the difference between the two different types of boundary conditions, i.e. Dirichlet and Neumann. In Chapter 4, the results are extended onto the nonstationary (evolutionary) problem. Since some of the analytical results are verified and illustrated with the help of numerical experiments, the last Chapter 5 provides details of the numerical methods used.

Although I quite like the thesis in general, I also have identified several points where presenting the material is slightly confusing and could be improved. The list of specific points is below.

Chapter 1

General comments

It would make the thesis easier to read if the equation(s) are shown earlier in the text, for instance moving the subsection 1.3.1 to as early as the beginning of section 1.2.

While Dirichlet boundary conditions are defined in (1.2.4)-(1.2.5), Neumann boundary conditions are never defined: the corresponding system is referred to as “close” but a mathematical expression is never given.

Some of the sections are badly missing references; for instance, there is not a single reference in sections 1.3.2 and 1.4.2.

Specific comments

Figure 1.3A is confusing, in fact not exactly correct. An ensemble of small patches will not necessarily result in the population extinction if the patches are located very close to each other (as seems to be the case shown in Fig.1.3A). In case of the non-zero Dirichlet conditions, the patches will then interact, effectively creating one large patch.

Definition 1.3 and (hence the definition of extinction regime given at the bottom of page 3) is slightly confusing. A positive v_{crit} only exists if it is imposed by the properties of the local population growth, i.e. the strong Allee effect. In the case of the logistic growth, $v_{crit}=0$, i.e. any small population will survive.

Chapter 2

General comment. As far as I understand, one of the main results of this chapter is the existence of the spectral gap. This is indeed an interesting property of the model. I am wondering if it might be possible to add a comment (or perhaps an example) to explain (show) what may be implications of such gap for the population dynamics in the ecological context. That would help to highlight the significance of this result.

Specific comments

Equations (2.0.1)-(2.0.2): In the formulation of kernel-based dispersal models, the kernel usually has the meaning of the probability density function, e.g. describing how likely an individual would travel to a given location in space. In this case, the integral of the kernel over entire space is exactly one (as the author mentions in the line after Eq.2.3.14 further down the text), and the integral over any finite domain is a number less than one. The right-hand side in Eq. (2.0.2) can be transformed as

$$\int_{\Omega} J(x-y) dy = \int_{\Omega} J(x-y) d(y-x) = - \int_{\Omega} J(z) dz \leq 1 \text{ (where } z=x-y\text{),}$$

so that $0 \leq \int_{\Omega} J(x-y) dy \leq 1$. It should be good to mention it in the text. Perhaps this observation might help to simplify some of the proofs? In particular, statement (2.1.1) then becomes a trivial consequence of the above.

Theorem 2.1.2:

(a) Domain Ω is said to be open (see Table 1.1). Correspondingly, due to the generalised Dirichlet conditions used in the form given by (1.2.4)-(1.2.5), apparently $v(x)=0$ on the domain boundary. However, Theorem 2.1.2 says that v_0 is strictly positive in domain Ω with a continuous extension onto domain boundary, hence apparently meaning that v_0 is positive also on the domain boundary. I am struggling to see how it can be, as it seems to contradict the pre-imposed boundary conditions (1.2.4)-(1.2.5).

(b) The phrase “the boundary value is not taken in the usual classical sense” is a bit obscure and confusing, some explanations are needed (and perhaps is important in the light of the previous comment).

Assumption 2.3.1: I am wondering if this assumption could be interpreted in some intuitive (or “visual”) way? Otherwise, it looks very formal and then a question may arise if it is realistic, i.e. do any kernel exist at all that satisfy it. Or perhaps giving an example of such kernel could be helpful.

Chapter 3

Specific comments

Page 35, Eqs. (3.0.2). Functions f and g are never discussed; meanwhile, depending on their properties, some generic properties of the system, such the existence of positive domain critical size or positive extinction threshold, may be different (e.g., depending on if the population growth is logistic or it is affected by the strong Allee effect).

Page 38, top: while it is correct to say that any root of function f corresponds to a spatially uniform stationary solutions, such solutions can be stable or unstable. In my opinion, the comments on top of page 38 are incomplete unless the issue of solution stability is mentioned.

Statement (3.2.3): $f(V^*)$ is just a number and so does f'' ; is it really necessary to consider their norms (and just their values)?

Page 43, Theorem 3.2.4, in particular the two lines after Eq. (3.2.27). I easily believe that the discontinuity of the solution may arise in the limit when ϵ goes to infinity. However, I am struggling to see how a discontinuity may emerge when ϵ is just “sufficiently large”. The latter apparently implies that there is a threshold value for ϵ – say ϵ^* – such as the solution is continuous for $\epsilon < \epsilon^*$ and becomes discontinuous for $\epsilon > \epsilon^*$. But this is very difficult to believe and indeed I can’t see such property is proven anywhere in the thesis. I suggest that the theorem statement should be corrected.

Figure 3.1. It would be helpful to add the values of p (as in Eq.3.4.1) that were used in simulation to the figure caption to explain what exactly is called “thin tail” and “fat tail”.

Figure 3.2. Is there any intuitive explanation why the critical size is (approximately?) constant for $p < 0.5$ but starts growing fast for $p > 0.5$? What is special about the value $p = 0.5$?

Figure 3.6. The corresponding value(s) of d_w should be added to the figure caption.

Chapter 4

Specific comments

Page 76, beginning of section 4.4.2. The phrase “population hovers exactly at the brink of survival” is difficult to understand and it is certainly not rigorous enough. Some necessary details should be added.

Chapter 5.

I don't have any concerns about this chapter.

In conclusion, I would like to emphasize that the above comments do not undermine the relevance and general scientific rigor of the work presented in the thesis. I have no doubt that the overall quality and the amount of work done by Maciej Tadej fully agree with the expectations for a PhD degree. It is therefore my opinion that Mr Maciej Tadej should be awarded PhD in Mathematics.

Sincerely,

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