Referee's report on the PhD thesis **On isomorphic and geometric properties of Banach spaces of continuous functions** *by Maciej Korpalski*

The thesis is devoted to various properties of Banach spaces of continuous functions, shortly $C(K)$-spaces, where K is a compact Hausdorff space. It consists of five chapters and an appendix. In Chapter 1 the content of thesis is briefly described. Chapter 2 contains some background knowledge and notation. The core of the thesis is formed by Chapters 3, 4 and 5. The results presented there come from five papers authored or co-authored by the applicant. Next I will give some comments to the individual chapters.

Chapter 1: This chapter contains a short introduction and a brief overview of main results. It is more or less well done. I have only two comments:

- In Theorem 1.0.1 it should be mentioned that the spaces with different value of α are not isomorphic.
- In Theorem 1.0.2 the assumption of compactness of K is missing.

Chapter 2: In this chapter some background information, notation etc. is collected. The choice of these things seems to be reasonable. However, there are some strange things, non-exact statements etc. More precisely:

- Page 6, lines 3–4: One-point compactification is defined only for locally compact spaces, not for all Tychonoff spaces.
- Page 6, paragraph before Fact 2.2.3: Compact lines are linear orders in which all nonempty subsets have both sup and inf. It is not enough that only one of the conditions holds (for example, in interval $(0,1]$ each nonempty subset admits sup, but not necessarily inf).
- Page 8, Lemma 2.3.1: Important property of $\widehat{T}$ is that is is one-to-one (for any T, regardless of surjectivity) and that its range coincides with the range of T.
- Page 8, Corollary 2.3.3: The condition is that the oscillation is zero at each $k \in K$.
- Page 8, lines 4–3 from below: Theorem 2.3.4 is not a generalization of Borsuk-Dugundji theorem, because it does not cover it. It is rather a variant, not generalization.

Chapter 3: This chapter is based on two papers – [6] (joint paper with A. Avilés, RACSAM 2024) and [53] (joint paper with G. Plebanek, Fund. Math. 2024). The chapter is focused on countable discrete extensions of compact spaces, mainly of compact lines. Note that L is a countable discrete extension of K, shortly $L \in CDE(K)$, if L is a compact space, $K \subset L$ is a closed subset and $L \setminus K$ is countable and discrete. Hence, we have a short exact sequence $0 \to c_0 \to C(L) \to C(K) \to 0$. The main question is whether this sequence splits, i.e., whether the respective copy of c_0 is $C(L)$ is complemented. More precisely, the question is for which K there is L such that the sequence does not split. This is studied using two properties – property $(\mathcal{R})$ says that there is a continuous retraction of L onto K and property $(\mathcal{E})$ says that there is a bounded extension operator $C(K) \to C(L)$. In this case $\eta(K, L)$ is the inf of the norms of such operators.

- Page 10, lines 6–8: The definition of property $(\mathcal{E})$ is strange, it should mean that 'there is an extension operator', not that 'there is no extension operator'.
- Lemma 3.3.1: This lemma is [53, Lemma 3.1] with a bit modified notation and proof. The modified proof is less understandable than the original one. More precisely: Let us consider the sentence 'Now, by (i), ...'. It is hardly understandable, the explanation in brackets is not helpful (while the original version has a bit different explanation, which is more clear). A correct explanation is the following: By (i) we know that for every $t \in [t_1, \to)$ we have $\nu_n[t, \to) = 0$ for almost all $n \in V \cap \omega$. Since $V \cap \omega$ is infinite and $\mathcal{U}$ is a free ultrafilter, we deduce that $\lim_{n \to \mathcal{U}} \nu_n[t, \to) = 0$.

- Theorem 3.3.2: It is completely correct, but the definition of x_n is unnecessarily complicated. In fact, definition of A_x is not useful, rather confusing. It serves only to define x_n. It would be much more reader-friendly to define simply $x_n = \inf\{x \in K; \mu_n^+(\leftarrow, x] \geq 1 - \delta\}$. It is, of course, equivalent, but it would eliminate unnecessary notation and hence simplify the situation.
- Theorem 3.3.3: Step 5 of the proof is not clearly explained (this applies also to [53, Theorem 3.3]). Note that in case elements s, t are fixed, no such decomposition is available. It is probably necessary to fix some $u \in (s, t)$ and set $I_1 = \{n \in F \cap \omega; x_n^0 \leq s < u \leq x_n^1 \leq x_n^2\}$ and $I_2 = \{n \in F \cap \omega; x_n^0 \leq x_n^1 \leq u < t \leq x_n^2\}$. Then $I = I_1 \cup I_2$ and sets F, H satisfy the assumptions also with respect to the pairs s, u and u, t. It took me some time to realize this.
- Section 3.4.1, real almost chains: This is, of course, a nice tool. But there are some things to be better explained:
  - It is not explicitly said whether the real almost chains are assumed to be strictly increasing, i.e., $A_x =^* A_y \Rightarrow x = y$. It seems that it is tacitly assumed, at least it is required for equality $w(K) = |X|$ in Lemma 3.4.3. (If not, the representation of K is degenerated a bit, it may be even a singleton.)
  - Under the 'strictly increasing assumption', K may be described just using X – using generalized intervals (sets of the form $[x, \infty) \cap X$ or $(x, \infty) \cap X$ for $x \in \mathbb{R}$) equipped with the reversed inclusion. This representation is more understandable than the abstract approach using ultrafilters.
- Theorem 3.4.8: This theorem is probably not directly in [6] or [53], it seems to be extracted from the proof of [6,Theorem 1.4]. The proof is probably correct, but not easy to follow. Some more explanation would be helpful. In particular:
  - It would be helpful to use the above description of K and a natural embedding of X into K ($x \mapsto [x, \infty) \cap X$). This is used tacitly in the proof in the definition of x_n^j.
  - On line 6 of the proof the 'order on $z \in K$' is addressed. But it seems that it is not an order on z (which is an ultrafilter) but an order on K (which is a set of ultrafilters).
  - It would be helpful to better explain the values of $\delta_x(B_y)$:
    One has $x < y \Rightarrow B_x \subseteq^* B_y \Leftrightarrow B_y/fin \in x \Leftrightarrow \delta_x(B_y) = 1$, while the converse to the first implication requires the 'strictly increasing assumption' mentioned above.
    This is important for the proof of the claim:
    Assume $x < x_n^0$: Then $n \notin B_x$, so $\delta_n(B_x) = 0$. Moreover, under the 'strictly increasing assumption' we have $\delta_{x_0}(B_x) = \delta_{x_1}(B_x) = \delta_{x_2}(B_x) = 0$.
    What if $x = x_n^0$: Note that $x_n^0 \in X$ only if the minimum is attained, so we get the same as above.
    Etc., note that the 'strictly increasing assumption' is probably needed.
- Construction 3.5.1: The construction is strange and not understandable, this applies also to [53, Construction 5.1] which is the same. Assume $x = (1, 0, 1, 0, 1, 0, \dots)$ and $y = (1, 1, 1, \dots)$. If I understand the definitions well, we have $x \prec y$. Moreover, $A_y = \{\emptyset, (1), (1, 1), (1, 1, 1), \dots\}$ and $A_x = \{\emptyset, (1), (1, 0, 1), (1, 0, 1, 0, 1), \dots\}$, so $A_y \setminus A_x = \{(1, 1), (1, 1, 1), \dots\}$ is an infinite set.

Chapter 4: This chapter is based on preprint [52], whose unique author is the applicant. The main object of interest are spaces of continuous functions on finite products of nonseparable compact lines and existence of onto isomorphisms, isomorphic inclusions and continuous linear surjections among such spaces. There are nice ideas, but simultaneously many strange things.

- Definition 4.2.3 and the preceding paragraph: The notion of 'disjoint union' is of course widely used, but normally not for sets, but for sets with some structure. It is not an operation, but rather a construction (for example, for topological spaces this is another name of a topological sum). In particular, the definition of $K^\uparrow$ is problematic: K_L and K_R are well-defined subsets of K. What is their disjoint union? Note that this cannot be canonically identified with a subset of K. Maybe one would like to say that the elements from the intersection $K_L \cap K_R$ are counted twice, but this is a nonsense in the standard set theory.

- Fact 4.2.4 is false. For example, let $K = [0, \omega_2]$, $\kappa = \omega$ and $k = \omega_2$.
- Corollary 4.2.5 is essentially correct, but it uses the problematic notation mentioned above.
- Proof of Theorem 4.3.2:

  Firstly, the proof of the first implication is finished by saying that 'the desired formula follows from Lemma 4.2.2'. This is not completely clear and requires more explanation. For example: Let μ and D be as in the text. By assumptions there is $f \in C(K)$ such that $\varphi|_{\{\delta_x; x\in D\}\cup\{\mu\}} = f|_{\{\delta_x; x\in D\}\cup\{\mu\}}$. Hence $f = g_\varphi$ on D. Since f is continuous and g_φ is κ-continuous, we get $f = g_\varphi$ on $\overline{D}$ (this uses Lemma 4.2.2) and hence $\varphi(\mu) = f(\mu) = \int f\, d\mu = \int g\, d\mu$.

  Further, in the proof of the converse, the term of 'support' is used. But 'support of a measure' is something else, the assumption here is not separable support (which is a uniquely defined set) but a separable set carrying the measure.
- Proof of Lemma 4.3.4 is not correct. More precisely: The lemma deals with two cases – isomorphic embeddings and surjections. The proof contains a common part and then the parts concerning the two cases.
  - The common part deals with an arbitrary operator $T : X \to Y$ between two Banach spaces. If $T^{**} : X^{**} \to Y^{**}$ is its second adjoint, then T is the restriction of T^{**} to X. Hence $T^{**}(X) \subset Y$. But implication $T^{**}(x^{**}) \in Y \Rightarrow x^{**} \in X$ is not true. Note that $T^{**}(X^{**}) \subset Y$ if and only if T is weakly compact. (For example, take $T : c_0 \to c_0$, $T((x_n)) = (x_n/n)$. Then $T^{**} : \ell^\infty \to \ell^\infty$ is given also by $T^{**}((x_n)) = (x_n/n)$. Hence $T^{**}(\ell^\infty) \subset c_0$ and, even, T^{**} is one-to-one.)

    However, the implication is valid if T is an isomorphic embedding. Indeed, in this case T^{**} is also an isomorphic embedding and also a w*-to-w* homeomorphism. (As any dual operator it is w*-to-w* continuous, so the restriction to the unit ball is a w*-to-w* homeomorphism. It follows that the inverse is also w*-to-w* continuous on the unit ball of the image, by Banach-Dieudonné theorem we easily deduce that it is w*-to-w* continuous on the whole image.)
  - In the common part it is correctly proved that $T^{**}(\kappa X) \subset \kappa Y$ (for any operator). Further, the operator $\widehat{T} : \kappa X \to \kappa Y/Y$ is correctly defined and it is observed that $X \subset \ker \widehat{T}$. So, by factorization we obtain an operator $U : \kappa X/X \to \kappa Y/Y$.
  - Let us look at the case of isomorphic inclusion. Then the proof is inspired by [20, Theorem 3.1]. But, the proof contains mistakes (fortunately they may be easily repaired). In this case U coincide with S from the paper, so it is one-to-one. Firstly, one has $dist(T^{**}\varphi, T^{**}X) \geq \frac{1}{\|T^{-1}\|}$, not $\geq \|T^{-1}\|$. Hence, the constants in the following computation should be changed. Further, in the final computation it is claimed without any explanation that $\varphi(T^*y^*) = 0$. This is true, because $T^*y^* = y^* \circ T = 0$ – it is really worth to mention explicitly that it follows from the assumption $y^*|T^{**}[X] = 0$.
  - The surjection case is not clear. If T is a surjection, then T^{**} is also a surjection (by the open mapping theorem together with Goldstine theorem). To prove that U is a surjection – which seems to be the claim, one needs to know that $\widehat{T}$ is a surjection. But this seems not to be obvious, one would need to know that $T^{**}(\kappa X) = \kappa Y$. Note that in [20] there is no analogue of the surjection case. In the text no argument is given, it is only claimed it is true.

    In fact, the surjection case is false. A counterexample is as follows. Let $X = C([0,1]^{\omega_1})$, $Y = C([0,\omega_1])$ and $\kappa = \omega$. Note that $[0,1]^{\omega_1}$ is separable, hence X^* is weak*-separable, so $\omega X = X$ and hence $SP_\omega X = \{0\}$. Further, it is easy to see (and follows from the text) that $SP_\omega Y \approx \mathbb{R}$. Finally, $[0,\omega_1]$ may be homeomorphically embedded into $[0,1]^{\omega_1}$, thus $X \twoheadrightarrow Y$.
  - The fact that the surjection case is false influence further results. Below I mention two important cases, I am not sure whether those two are the only ones. In [20] there is no surjection case (as mentioned above), but there is the case of surjective isomorphism. Note that the case of surjective isomorphism can be easily transferred to the current situation.

- Proposition 4.3.8: The surjection case is false. As indicated above, if $Y = C([0,1]^{\omega_1})$, $X = C([0,\omega_1])$ and $\kappa = \omega$, then $Y \twoheadrightarrow X$, $sp_\omega(Y) = 0$ and $sp_\omega(X) = 1$.
- Proposition 4.3.9 may be formulated using supremum.
- Page 35, proof of Lemma 4.4.2, the second claim: Note that $K \times \{0\}$ is dense in $\mathbb{K}$, hence $\|Sf\| = \|f\|$ for each $f \in C(\mathbb{K})$, hence S is an isometry. So, the only thing to be proved is surjectivity of S.
- Page 36, line 4 from below: Theorem 4.4.3 does not exist, it is a lemma.
- Page 37, proof of Lemma 4.4.6: Formula (4.4.1) is referred to only as (1) (three lines later).
- Page 38, lines 1 and 3: Formulas for t_0 are started with 'We know that' without an explanation. This is probably quite easy, anyway a reader would expect some argument.
- Corollary 4.5.3: The proof of the surjection case is missing because the respective part of Proposition 4.3.8 is false. However, the counterexample disproving Proposition 4.3.8 does not work here, so it is not clear whether the statement is true or not. It is possible that the surjection cases of Lemma 4.3.4 and Proposition 4.3.8 are true in some special situations, for example for spaces of continuous functions on finite product of compact lines. But in such a case a proof should use special structure of the spaces and it is not clear to me how to do that.
- Lemma 4.5.6 and its proof:
  - In the statement it should be 'c_0-sum of spaces $C(K)$ for $K \in \mathcal{C}_\kappa^{n-1}$'.
  - In the definition of $J(k,b)$ one should write $\{(k,b)\}$ instead of (k,b).
  - The use of $K_i^\uparrow$ is problematic (but understandable) as explained above.
  - It sounds strange if it is said that it is standard to check that S is a continuous linear operator and only then a proof that S is well defined starts. A correct formulation is that S is clearly a bounded linear operator to the product of respective ℓ_∞ spaces and that it will be proved that the range is in the product of c_0-spaces.
  - The arguments using oscillation are not understandable. If $g \in C(\mathbb{K})$, then $g|_{\widetilde{K}}$ is continuous and hence its oscillation is zero. Maybe $\widetilde{K}$ should be considered with the topology of K (there is a natural bijection of $\widetilde{K}$ and K) rather than with the topology inherited from $\mathbb{K}$.
- Proposition 4.6.1: In the proof, 'hyperspaces' should read 'hyperplanes'.
- Question 1 on p. 42: Note that $C([0,\omega^\omega])$ does not embed into $c_0(\Gamma)$ for any set Γ. The reason is that each subspace of $c_0(\Gamma)$ has the Dunford-Pettis property but there is a subspace of $C([0,\omega^\omega])$ without the Dunford-Pettis property. (A stronger version is addressed in [Kalenda, Spurný: On quantitative Schur and Dunford-Pettis properties. Bull. Austral. Math. Soc. 91 (2015), no. 3, 471-486], but the results itself should be much older.) Therefore, Proposition 4.6.2 provides a negative answer to Question 1.
- Proof of Proposition 4.6.2: The work with sets A_x is not very clear. It would be more understandable and more canonical to proceed 'conversely'. More precisely: For $f \in D$ and $\varepsilon > 0$ define $A_{f,\varepsilon} = \{x \in (0,1); \exists k \in K : |f((x,0),k) - f((x,1),k)| > \varepsilon\}$. This set is necessarily finite. So, $A_f = \{x \in (0,1); \exists k \in K : f((x,0),k) \neq f((x,1),k)\}$ is countable. So, we may find $x \in (0,1) \setminus \bigcup_{f \in D} A_f$ and proceed.

  Then C_x is an isometric copy of $C(K)$ in $C(\mathbb{S} \times K)$ and the quotient map is an isomorphism on this subspace.

Chapter 5: This chapter is based on preprints [54,55], a joint work with the supervisor. It concerns Banach-Mazur distance between some $C(K)$-spaces. Some improvements of known estimates are given. The first part of results deals with countable compact spaces, the second one with non-scattered compact spaces. The main methods include some new clever tricks together with some elementary computation, with some parts computer-assisted. This makes some of the results hard to check.

- Lemma 5.2.2: The term '1-norming' is misleading. Usually, if X is a Banach space, a subset $A \subset X^*$ is called 1-norming if $A \subset B_{X^*}$ and $\|x\| = \sup_{x^* \in A} |x^*(x)|$ for each $x \in X$. Or, a subspace of X^* is 1-norming if its intersection with B_{X^*} is 1-norming according to the previous sentence. Note that measures ν_y are not necessarily in the unit ball. Moreover, it seems that here '1-norming' means that $\sup_{y\in L} |\nu_y(g)| \geq \|g\|$ for $g \in C(K)$. This is, surely, an important property, but it is not '1-norming' in the usual sense.

- Lemma 5.2.4: This is a variant of [55, Lemma 2.3], but both the assumptions and the conclusion are a bit different. I have not checked the variant of [55], but the current version seems not to be completely correct:
  - Firstly, it is claimed that $\|\psi_0\| \leq \varepsilon$ by (iii). Note that it is probably necessary to use also (ii). An explanation should be given. (By (iii) we have $|\psi| \leq r$ and $|\psi \pm \varphi| \leq r$. By (ii) we have $\varphi \geq r - \varepsilon$ on A. It follows that $|\psi| \leq \varepsilon$ on A.)
  - Further, in (5.2.1) one gets $\nu(\psi_1) \geq \nu(\psi) - t\varepsilon$. It is not clear to me how to get $1 - t\varepsilon$.
  - It is claimed that in the computation on the last line of p. 48 inequalities (5.2.1) and (5.2.2) are used. However, the first inequality uses (iii) without saying and in the second one (5.2.2) is used. (5.2.1) is not used at all. Moreover, there seem to be a computational error, $-\nu(\varphi)$ should read $-\frac{\nu(\varphi)}{r}$.
  - On the first line of p. 49, the correct inequality should be $\nu(\psi_1) \geq \nu(\psi) - t\varepsilon$. Then one gets the inequality on the second line, with $\nu(\varphi)$ replaced by $\frac{\nu(\varphi)}{r}$.

  Summarizing: (5.2.1) is not correct and not used later and the inequality really proved is slightly different. This, of course, influence further results.
- The first line of 5.3.1: K should be moreover countable.
- In the proof of Lemma 5.3.1, to deduce $\|\varphi\| \leq 1$ one needs to use that T is norm-increasing. It would be better to say that.
- The notation $a \lesssim b$ is strange. It is defined using inequality of limits. Does it mean that the limits are assumed to exist? Should they be finite? Nonzero? Further, it is defined for functions on a right-neighborhood of zero, but it seems it is used also for sequences. Note, that the usual meaning of such asymptotic inequalities is different – it is used for non-negative functions in the sense that there is some constant $c \in (0, \infty)$ such that $a \leq cb$ on a right neighborhood of zero (and analogously for sequences).
- Lemma 5.3.2 and its proof:
  - The role of t is very limited. It would be enough to prove it for $t = s$.
  - The 'approximate calculations' of p. 50 have unclear meaning. Well, we have a sequence $(i(n))$ such that $|\nu_{i(n)}(g_n)| \geq \|g_n\| \geq 1$. But what means that $|\nu_{i(n)}(g_n)| \lesssim \frac{|\nu(f)|}{s} + |\nu_{i(n)}(e_n)|(1-\frac{1}{s})$? I assume that the definition is applied to sequences. But why $\lim |\nu_{i(n)}(g_n)|$ should exist? Why the limit of the right-hand side should exist? Why the inequality holds? I assume it should come from the definition of g_n – we get

$$|\nu_{i(n)}(g_n)| \leq \frac{|\nu(f)|}{s+\varepsilon} + |\nu_{i(n)}(e_n)|(1 - \frac{1-\varepsilon}{s+\varepsilon}).$$

    Well, one may study not the dependence on n for given ε, but dependence on ε for given n. But this is complete mess. OK, n is fixed and $g_n = g_{n,\varepsilon}$. But $i(n)$ depends on ε (it depends on δ which depends on ε). So, neither existence of limits with respect to $\varepsilon \to 0+$ is not clear. But note that, if we do not use the hardly understandable notation $\lesssim$, we get an exact inequality

$$|\nu_{i(n)}(e_n)| \geq \frac{1 - \frac{|\nu(f)|}{s+\varepsilon}}{1 - \frac{1-\varepsilon}{s+\varepsilon}} = \frac{s + \varepsilon - |\nu(f)|}{s + 2\varepsilon - 1} > 0$$

    for each $\varepsilon > 0$ (as $|\nu(f)| = \lim_i |\nu_i(f)| \leq \|Tf\| = s$, $\varepsilon > 0$ and $s > 1$), so $(i(n))$ is unbounded for each $\varepsilon > 0$. (While it is not clear whether $|\nu(f)| < s$.) So, the use of $\lesssim$ is not needed and it is rather confusing.
  - Lemma 5.2.3(a) is then invoked to find function h. But this is not correct. By the quoted lemma we have $|\nu_i|(K \setminus C) > |\nu|(K \setminus C) - \varepsilon$ for i large enough. So, for i large enough there is a norm-one function h_i supported outside C with $\nu_i(h_i) > |\nu|(K \setminus C) - \varepsilon$. Hence the function h depends on i.

However, one may find h independent on i, not using the lemma, but rather repeating the proof. We find h such that $\nu(h) > |\nu|(K \setminus C) - \varepsilon$. Now, since $\nu_i(h) \to \nu(h)$, $\nu_i(h) > |\nu|(K \setminus C) - \varepsilon$ for i large enough.

– The final application of Lemma 5.2.4 is not clear at all. It is said what is r, A, φ, ψ. Then assumptions $(ii), (iii)$ from Lemma 5.2.4 are fulfilled. But it is not said what are ν and t. Is it applied to ν or to $\nu_{i(n)}$? Note that there seems not to be any guarantee that t from the current lemma satisfies (i) from Lemma 5.2.4 for ν or $\nu_{i(n)}$. So, perhaps one may apply it for $t = |\nu|(K)$ (or $t = |\nu_{i(n)}|(K)$). By applying to $\nu_{i(n)}$ we get (since $r = 1$, the error mentioned above is not relevant)

$$|\nu_{i(n)}|(K) \geq 2|\nu_{i(n)}(e_n)| - \nu_{i(n)}(f\chi_C) + |\nu_{i(n)}(h)| - 2\varepsilon|\nu_{i(n)}|(K)$$

I see no way how to deduce something similar to the inequality in the text. Firstly, on the left-hand side there is $|\nu_{i(n)}|(C)$ instead of $|\nu_{i(n)}|(K)$. Note that Lemma 5.2.4 cannot be applied to C in place of K, as $\psi = h$ is supported outside C. Further, I cannot imagine where the second and third terms on the right-hand side come from and how the conclusion is done.

- In section 5.3.2 statements of uncertain meaning are repeatedly used, like $\nu_1(A) \approx 0$ etc. This makes the text unreadable and uncheckable. I give up. I skip also Sections 5.3.3 and 5.3.4.
- Proof of Theorem 5.4.1: Is Φ a family ... (i.e., some family containing functions of the given form) or the family ... (i.e., the family of all functions of the given form)?

Overall comments: The thesis is based on two published papers and three preprints, the unifying topic is the structure of $C(K)$ spaces. The two published papers (with co-authors) provide content of Chapter 3. This chapter is mostly nice, most of the comments concern presentation which may be better. The only exception is Construction 3.5.1 which seems to be strange and not understandable. Chapter 4 is inspired by a preprint (whose unique author is the applicant). It presents a nice theory with interesting applications. Some parts of notation ($K^{\uparrow}$) are problematic, but it is understandable and may be correctly interpreted. However, the key tool – Lemma 4.3.4 – is not correctly proved and, moreover, one half of the lemma is false. As a result, the respective part of Proposition 4.3.8 is also false and the proof of the respective part of Corollary 4.5.3 is missing. Chapter 5 is inspired by two preprints (co-authored by the supervisor). Section 5.4 inspired by one of the preprints is nice and interesting. Sections 5.2 and 5.3 (inspired by the last preprint) probably contain nice ideas, but they are hardly readable. The main reason is the use of unclear notation $\lesssim$ and $\approx$ with uncertain meaning. The work with this notation seems to be impossible to check as it does not follow basic rules of writing a mathematical text.

Conclusion: The thesis contains new nontrivial interesting results and due to its content has potential to be outstanding. However, the presentation has some important deficiencies. The thesis contains material from five papers (two published ones and three preprints). The presentation of three papers out of five is more or less nice (with only one exception – Construction 3.5.1, see above). One of the remaining papers contains an essential error (one half of the key lemma (Lemma 4.3.4) is false, hence some results are doubtful). The last paper uses strange notation in a way which does not follow the rules of mathematical work. This makes it unreadable and uncheckable (although this part seems to contain nice ideas and potentially nice results). Anyway, I think that already the content of the three correct papers is sufficient for awarding of the scientific degree of doctor. Further, the correct part of the fourth paper is also valuable. Therefore I recommend to award the scientific degree of doctor. The above-mentioned errors and problems should be addressed and explained during the defense.

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