PhD thesis review of '**Asymptotic behavior of the extremal position in a branching random walk**' by Krzysztof Kowalski

The present thesis is concerned with multi-type branching random walks and perturbed branching random walks. Arguably, the branching random walk and the branching Brownian motion are among the most popular objects of research within a branching processes community. Having a general point process as the input process, the branching random walks can be used to model population dynamics for a wide range of populations. The one-dimensional (standard) branching random walks are rather well-understood. On the other hand, there are many open problems pertaining to multi-type branching random walks, let alone, perturbed branching random walks that were introduced only recently. Some of these are tackled in the present thesis.

The thesis is structured as follows. The introductory Chapter 1 provides a motivation and briefly discusses some relevant earlier works.

Chapter 2 explores irreducible multi-type branching random walks. More precisely, it is assumed that some power of the mean matrix has positive entries only, which ensures irreducibility of the process. Theorem 2.1 asserts that the maximal position in the nth generation, properly normalized without centering, converges in distribution to a Weibull distributed random variable with a random parameter. Here, it is assumed that the displacements of individuals relative to the mother's position are independent and identically distributed. For mothers of different types, the displacements may have different distributions, with right distribution tails being regularly varying at infinity and left distribution tails satisfying a mild growth condition. Theorem 2.1 extends the ideas of Durrett (1983) (one type of particles) and Bhattacharya et al. (2019) (many types of particles; leaves in the underlying tree are not allowed). Under the assumption that the right distribution tails of displacements are semi-exponential, rather than regularly varying, Theorem 2.5 states that the maximal position in the nth generation, properly normalized without centering, converges almost surely to a deterministic constant. Theorem 2.5 generalizes a result of Gantert (2000) obtained for branching random walks with one type of particles.

Section 3 investigates reducible multi-type branching random walks. Theorems 3.1 and 3.5, the main results of Section 3, are counterparts of Theorems 2.1 and 2.5 obtained under comparable assumptions, which had to be adjusted to take into account reducibility. Obviously, the argument worked out in Section 3 is more delicate than that in Section 2.

Section 4 focuses on one-dimensional perturbed branching random walk, in which each position of the underlying standard branching random walk gets an independent perturbation of specific form. As it is always the case for perturbed models, the main question is how does the presence of perturbations affect properties of the unperturbed model. For standard branching random walks, it was shown by Biggins (1976) that, under mild assumptions, the maximal position in the nth generation, normalized by n, converges almost surely to a deterministic constant. Aïdékon (2013) obtained a result exhibiting the rate of distributional convergence in Biggins' limit theorem. Theorem 4.1 (4.2) provides a condition imposed on perturbations that ensures that the almost sure asymptotic behavior of the maximal position in the nth generation of the perturbed model is different from (mimics) that in Biggins' theorem. Theorems 4.3, 4.4 and 4.5 are more complicated results. They investigate in three different regimes the rate of distributional convergence in the aforementioned almost sure limit theorems. Although these theorems bear a resemblance to Aïdékon (2013) result, the limit distributions in the present thesis are essentially different.

Comments and some minor points given below may be used to obtain a few cosmetic improvements of the thesis.

COMMENTS:

1. Remark 2.2 on p. 16. The formula $a_n = L^{\#}(\rho^{n/r})\rho^{n/r}$ is incorrect. For instance, if $L_I(x) = \log x$ and $r = 1/2$, then $a_n = 4\rho^{2n}(\log \rho^n)^2$, whereas $L^{\#}(x) = 1/\log x$. The correct formula is $a_n = (\ell^{\#}(\rho^n))^{1/\rho}\rho^{n/r}$, where $\ell^{\#}$ is the de Bruijn conjugate of $x \mapsto 1/L_I(x^{1/r})$, see, for instance, Proposition 1.5.15 in the book Bingham et al. (1987) on regular variation. A similar correction is also needed in Remarks 2.6, 3.2 and 3.6.

2. Since μ in Section 4 is the distribution of a positive random variable, condition (H) ensures that μ belongs to the domain of attraction of a γ-stable distribution *concentrated on the positive halfline*. Thus, using the characteristic function as in (4.1) rather than a Laplace transform is misleading and complicates things significantly.

3. The equality $\mathbb{P}\{H_\theta = 0\} = 0$ on p. 57 is elementary and does not require the proof involving characteristic functions. Indeed, the distribution of H_θ is stable with a random parameter. Since the stable distribution is (absolutely) continuous, conditioning on the parameter does the job.

MINOR POINTS:

p. 7, l. -14: 'branch at exponential times' sounds misleading. The first branching time has indeed the exponential distribution of unit mean. However, this is not the case for the other branching times.

p. 8, l. 5: ϕ has not been introduced so far. Perhaps, it is better to move this sentence at the very end of Section 1.1.

p. 8, l. 6: A gap is needed before 'and'; Selke $\to$ Sellke

p. 8, l. 8 and 10: $n \to \infty$ should be $t \to \infty$

p. 8, l. -11: 'the number of offspring' sounds better

p. 8, l. -6: 'The process continues infinitely' sounds misleading. If $\mathbb{E}[N] < 1$, then the process dies out in a finite time. One needs here the supercriticality assumption $\mathbb{E}[N] > 1$ ensuring the survival with positive probability. This remark also applies to p. 11, l. -15.

p. 8, l. -2: Does the author mean $\mathcal{Z}$ rather than ξ?

p. 9, l. -13: 'most right' $\to$ 'right-most'

p. 9, formula (1.1): $n \to N$

p. 10, l. -15: There were numerous, more recent publications on branching random walks.

p. 10, formula (1.3): The variable ξ seems to have not been defined so far. Also, one should say here that $\xi_1, \xi_2, \dots$ are assumed independent and identically distributed.

p. 13, l.-9: The word 'real' can be safely removed.

p. 13, l. -4: $X_n \to X_v$

p. 16, Theorem 2.1: q has to be replaced with r

p. 25, Thm 2.5: It is worth recalling that ρ is the principal eigenvalue of the mean matrix M.

p. 26, l. 10: its $\to$ it is

p. 26, l. -2: better 'impose' in place of 'make'

p. 27, l. -8: the Lemma 2.8 $\to$ Lemma 2.8

p. 29, l. -13: better 'Theorem 2.5' rather than 'the Theorem'

p. 32, l. 9: Either replace the fullstop with a comma, or start a new sentence with 'This is a straightforward...'.

p. 33, l. -1: Should ρ_a be $\rho(a)$?

p. 51, l. -10: The quantity R_n^* does not seem to have been defined.

p. 51, l. -9: 'Weak centered asymptotics' sounds vague.

p. 52, l. 6: The line starts at a wrong place.

p. 53, l. 45: The notation W_θ^∞ is unfortunate. Perhaps, this is a typo, and the author meant $W_\infty(\theta)$. This remark also concerns Theorem 4.3 and elsewhere.

p. 54, l. -4 and p. 55, l. 4: Better 'mean' rather than 'intensity'.

p. 57, l. 5: The formula has to be moved to the next line.

Section 4 is based on the (single-authored) article by Krzysztof Kowalski 'Branching random walks with regularly varying perturbations' published in ESAIM in 2024. According to Mr. Kowalski's CV, the results of the thesis were discussed at several conferences and workshops. Summarizing, I believe sufficient work has been done towards presenting Mr. Kowalski's results to the scientific community.

The results of Mr. Kowalski are an important step towards further understanding of subtleties of branching random walks. In general, the thesis is a nice piece of mathematical work. When proving his results Mr. Kowalski skillfully uses advanced probabilistic techniques and tools and works out new arguments. The thesis is thoughtfully organized and reasonably well-written. It was a pleasure to read it. At places where some heavy analytic machinery has been expected, the author rather offers attractive probabilistic arguments. I have no doubts in recommending to accept this PhD thesis and award Krzysztof Kowalski the scientific degree of doctor.

Head of Operations Research Department,
Taras Shevchenko National University of Kyiv,

Prof Dr Oleksandr Iksanov