WWU MÜNSTER

Fachbereich 10
Institut für Mathematische
Logik und
Grundlagenforschung

WWU Münster | Institut für Mathematische Logik und Grundlagenforschung | Einsteinstr. 62 | D–48149 Münster | Germany

Prof. Dr. Martin Hils

Einsteinstr. 62
D-48149 Münster
Germany

Tel. +49 251 83–32685
hils@uni-muenster.de

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Report on the doctoral thesis

Theory of models with group actions, with special discourse on theories of fields

by Daniel Max Hoffmann

It is a recurrent theme in model theory to study expansions of structures which are well-understood from a logical point of view. A notable example, starting with work of Lascar, is the study of models of a (stable) theory T, which are equipped with a distinguished automorphism named in the expansion. Around 1990, van den Dries, Macintyre and Wood realized that in the case where T=ACF is the theory of algebraically closed fields, the corresponding theory admits a model companion, ACFA, which is the theory of existentially closed difference fields. The fixed field of a model of ACFA is perfect, pseudo-algebraically closed (PAC), with absolute Galois group $\widehat{\mathbb{Z}}$, i.e., by work of Ax, it is a pseudofinite field.

Chatzidakis and Hrushovski studied the theory ACFA from the point of view of geometric model theory. They showed in particular that (i) ACFA is (super-)simple, (ii) that non-forking independence admits a description in terms of algebraic independence – so in terms of non-forking in the reduct ACF – and that (iii) ACFA eliminates imaginaries, i.e., all definable quotients are isomorphic to definable sets. The model theory of ACFA has had important applications to number theory (Hrushovski's proof of the Manin-Mumford conjecture) and to algebraic dynamics, the main model-theoretic result used being Zilber's trichotomy principle which is true in ACFA. The most spectacular result, due to Hrushovski, states that ACFA equals the theory of the non-standard Frobenius automorphism acting on an algebraically closed field. Moreover, ACFA has played an important role in the abstract development of simplicity theory.

In work of Chatzidakis and Pillay, many basic results have been extended from ACFA to the model companion of the theory of models of a stable theory T equipped with an automorphism, denoted by TA if it exists, e.g., (i) and (ii) above. As for (iii), a weak version of it, termed geometric elimination of imaginaries, holds in TA whenever the original theory T eliminates imaginaries.

For many stable theories T, TA is known to exist, e.g., for differentially closed fields of characteristic 0 or separably closed fields. In general, it is a rather subtle issue whether TA exists for a given stable theory T. It is an elementary fact, due to Kudaiberganov, that if TA exists, then T does not have the finite cover property. Baldwin and Shelah gave a complete characterization, in terms of properties of T, of when TA exists, but this characterization is difficult to use in concrete situations. In the important special case of a strongly minimal theory T, by a result of Hasson and Hrushovski, TA exists if and only if T has DMP, i.e., Morley degree is definable in T.

The dissertation of Daniel Hoffmann deals with a variation on this theme. Given a group G and a theory T, he studies models of T equipped with an action of G by automorphisms. The case mentioned above thus corresponds to $G = \mathbb{Z}$. For free groups, this had already been studied previously in some cases. For T the theory of fields, by a result of Hrushovski, for two commuting automorphisms (i.e., for $G = \mathbb{Z} \times \mathbb{Z}$), no model companion exists. Sjögren (unpublished, 2005) has studied the model theory of fields with an action of a group G, for arbitrary G. Below, we will discuss the relationship between parts of Hoffmann's dissertation and Sjögren's work.

In Chapter 2 of his dissertation, Hoffmann undertakes a general study of models of T equipped with a G-action. The corresponding theory is denoted by T_G. The general assumptions are that T is inductive and admits a model companion T^{mc} which eliminates quantifiers and imaginaries, and that T_G admits a model companion T_G^{mc}. Hoffmann observes that already in the case of fields, it may happen (e.g., when G is finite non-trivial, discussed in Chapter 3) that an existentially closed (e.c.) model of T_G is not an e.c. model of T, and so one may not reduce to the case where $T = T^{mc}$. Moreover, he gives some elementary examples of unstable theories T, where T_G^{mc} exists for finite G (dense linear orders, random graph, atomless boolean algebras), contrary to the case $G = \mathbb{Z}$, where no unstable example is known.

Guided by what happens in the case of fields, Hoffmann calls an extension $A \subseteq B$ of definably closed substructures of a model of T^{mc} *regular* if A is relatively algebraically closed in B, and he introduces the notion of a *PAC substructure* of a (large) model $\mathfrak{D}$ of T^{mc}, as one which is e.c. in every regular extension inside $\mathfrak{D}$. The relationship with other notions of a PAC substructure (due to Hrushovski and Pillay-Polkowska, respectively) is discussed. Hoffmann then proves some general preliminary results around the Galois theory (from the point of view of T^{mc}) of models of T_G, establishing in particular a very useful characterization of when a G-action may be extended to a proper algebraic extension (Lemma 2.2.27). From then on, T^{mc} is assumed to be stable.

We assume now that $\widetilde{M} = (M, \sigma_g)_{g \in G}$ is a model of T_G^{mc}, with subset of invariants M^G, and denote by N the algebraic closure of M^G in the sense of T^{mc}. Hoffmann makes the interesting observation that the canonical epimorphism $Aut(N/M^G) \to Aut(N \cap M/M^G)$ of profinite groups is a Frattini cover, and even the univeral Frattini cover in case T is the theory of fields and G is finite (proved in Chapter 3). He shows that M is a PAC substructure of the ambient model of T^{mc}. If G is finitely generated, in which case M^G is definable in $\widetilde{M}$, he proves that M^G is PAC as well, and moreover a bounded substructure, i.e., the size of the Galois group (in the sense of T^{mc}) of M is bounded independently of the model $\widetilde{M}$. Simplicity of M^G then follows by a theorem of Pillay-Polkowska.

As main results, Hoffmann shows, assuming a technical condition, that (i), (ii) and (iii) from above all hold in T_G^{mc}: the theory T_G^{mc} is simple, with forking given by T^{mc}-forking of the corresponding G-orbits (Theorem 2.3.22), and T_G^{mc} has geometric elimination of imaginaries, i.e., every imaginary element is interalgebraic with a real tuple (Theorem 2.3.35). The proof techniques for these results are similar to the case where $G = \mathbb{Z}$, but some additional twists are needed, due to the fact that M is in general not a model of T^{mc}. The technical condition is shown to hold in the important case where M is bounded, and Hoffmann proves that this is the case e.g. for G finite.

Chapter 3 of the thesis is mainly based on the article *Existentially closed fields with finite group actions* (joint with Kowalski, accepted for publication in J. Math. Log.), and it is devoted to a study of the case where T is the theory of fields and G is a finite group. With the help of geometric axioms, in the spirit of the geometric axioms for ACFA, Hoffmann introduces a first-order theory G-TCF in the ring language augmented by unary function symbols for the elements of G. The main result of the chapter is that G-TCF equals T_G^{mc} (Theorem 3.1.10), which shows in particular that the model companion T_G^{mc} exists and thus that the abstract results proved in chapter 2 apply in this case. Improving on (iii), Hoffmann shows that G-TCF eliminates imaginaries, once finitely many elements are named with constants. Hoffmann establishes the following interesting algebraic characterizations of models of G-TCF: if G acts faithfully on a perfect field K, then the structure $(K, \sigma_g, g \in G)$ is a model of G-TCF if and only if the subfield of invariants K^G is PAC and the G-action may not be extended to any proper algebraic extension of K; moreover, this is equivalent to the condition that any K-irreducible variety defined over K^G has a K^G-rational point. Furthermore, in case G is (finite) cyclic, Hoffmann exhibits an explicit example of a model of G-TCF in characteristic $p > 0$, with underlying field a subfield of $\mathbb{F}_p^{alg}$.

In a short appendix, Hoffmann sketches generalizations of the results of Chapter 3 to finite group scheme actions which are contained in the article *Existentially closed fields with G-derivations* (published in the J. Lond. Math. Soc.) which is joint with Kowalski.

The results obtained in the thesis are strong and satisfactory, and they apply potentially to many situations. Hoffmann clearly shows that he masters very well the technical material, both from model theory and from Galois theory. In his work, Hoffmann transposes various concepts from field theory to arbitrary stable theories in a very nice and fruitful manner.

The dissertation is mostly well written, and the proofs of the main results are all correct. However, in Chapter 2, there are several inaccuracies: the discussion of the problem of characterising those stable theories T for which TA exists is not adequate (nfcp is a rather obvious necessary condition, but the property characterising existence of TA is much more subtle); in Example 2.1.3, the description of types and the stability spectrum are incorrect as stated; moreover, there are several issues with Example 2.1.8. I have added to my report a separate list containing details about these, as well as minor issues and some suggestions on how to improve the presentation.

Some words of critique concerning Hoffmann's blanket assumption that T_G^{mc} exists are in order. Large parts of the material in Chapter 2 could have been developed without this assumption, working instead with the category of e.c. models. (This is briefly mentioned on page 10 of the dissertation, but not further pursued.) Moreover, it would have been natural to discuss existence of T_G^{mc} for G finite, in the case of other theories T, e.g., for differential fields in characteristic 0.

As I have mentioned at the beginning of my report, fields with G-actions have already been studied previously by Sjögren in his PhD thesis in 2005, but his results have never been published in a peer-reviewed journal. In the introduction of their paper *Existentially closed fields with finite group actions*, Hoffmann and Kowalski state that they had not been aware of this work, and so it is not surprising that there is a rather large overlap between Sjögren's work and the results of Chapter 3 in Hoffmann's dissertation. Contrary to Hoffmann, Sjögren does not use the blanket assumption that T_G^{mc} exists but rather works in the category of e.c. models. Some of Sjögren's results are more general than the corresponding results of Hoffmann, e.g., he shows that for arbitrary G, if $(K, \sigma_g)_{g \in G}$ is e.c., then both K and K^G are PAC; and that for G finitely presented, the absolute Galois group of K^G is the universal Frattini cover of the profinite completion of G.

To conclude, Daniel Hoffmann constitutes a very valuable contribution to the model theory of structures with automorphisms. The work contains strong results both in the general framework and in the particular case of fields. The overlap with Sjögren's work is unfortunate, but since the results were obtained completely independently, this fact does not weigh too much. I recommend that this thesis be accepted and that Daniel Hoffmann be granted a doctoral degree.

Yours sincerely,

Martin Hils

List of comments/suggestions (2^7 refers to page 2, line 7, 2_7 to page 2, line 7 from the bottom).

- 2^{10}: In the definition of κ-saturation, the parameter sets should be of cardinality $< \kappa$.
- page 2, 2 lines before 1.2.1: It should be "... every $M' \models T$ such that $M \subseteq M'$, it ..."
- 3^{13}: It should be "... p_0 is consistent..." instead of "... q is consistent..."
- 4_4: You should add that you assume $char(K) = p > 0$ when defining the perfect closure.
- Example 2.1.3: In the middle of page 12 it should be $M \cup (\prod_{H<G} G/H)^\omega$, where H runs over finitely generated subgroups of G, as you informed me already some weeks ago (in an email by Piotr Kowalski). Moreover, there are several other mistakes in this example. The type of an element $x \notin G \cdot A$ is not unique. One also needs to specify the stabilizer subgroup G_x in order to get a complete type, since the G-action is not free. Consequently, the cardinality statement about the type space $S_1(A)$ is wrong as well. What one gets is that for $\lambda \geq 2^{|G|} + \aleph_0$, T_G^{mc} is λ-stable. Also, when the example is taken up again (in 2.3.25) you claim that T_G^{mc} eliminates imaginaries. But this is not the case. If you add to T^{mc} for the "empty" theory T sorts for finite subsets, forcing elimination of imaginaries for T^{mc}, then T_G^{mc} should eliminate imaginaries, though, in the same sorts.
- Example 2.1.5: An easier way to present this would be a Fraïssé amalgamation: The class $\mathcal{C}$ of finite graphs with a G-action is a Fraïssé class (i.e., has (HP), (JEP) and (AP)), and it is uniformly locally finite. So the theory of the Fraïssé limit of $\mathcal{C}$ is ω-categorical, has QE, and it also follows that it is the model companion of the theory of graphs with a G-action.
- 15_{13}: I don't understand what you mean by "another additional axiom such as: $x^d = 1$ or $x^d = 0$" (for some $d > 0$), as in both cases, in the class of unitary rings, only the 0-Ring (where 0=1) satisfies the relevant axiom for all x.
- 15_{8+7}: The ring $R[t_1, \ldots, t_e]$ is in general not of exponent d, as is shown by the following counter-example: $d = 3$, $R = \mathbb{F}_2$ (which is a ring of exponent $d = 3$), $G = \{1\}$, so $e = 1$ and $R[t] = \mathbb{F}_2[X]/(X - X^3)$. Then $(1+t)^3 = 1 + t + t^2 + t^3 = 1 + t^2 \neq 1 + t$, since clearly $t \neq t^2$ in $R[t]$. The construction works for $d = 2$, so the example of Boolean algebras which follows in Remark 2.1.9 is ok.
- In Example 2.1.10, you may want to mention that G is finite.
- The regularity condition corresponds to stationarity (as mentioned in Remark 2.2.2(1), already considered in previous work by Hrushovski et al.
- 19^{18}: It should be three times $dcl_{\mathcal{L}}^{\mathfrak{D}}(N)$ instead of N.
- 19_3: You mean Definition 2.2.4 here and not Definition 2.2.5.
- Your definition (2.2.1) of a PAC substructure (PAC$_{reg}$) will in general not be preserved by $\equiv$. It might be interesting to discuss this. On the other hand, in the cases where you prove that PAC$_{reg}$ holds (2.2.45 and 2.2.47), actually something stronger holds, since you know that you have κ-saturated models of the relevant theory which are PAC.
- 20^{10}: In order to make it work, you need not only $\mathrm{mlt}(\psi) = 1$, but also $RM(\psi) = RM(p)$.
- Remark 2.2.17: Typo: It should be B_i instead of M_i in part 1 (for $i = 1, 2$).
- In definition 2.2.39, you should require that π is an *epimorphism*.
- 31_7: It should be $\hat{\sigma}_g \in Aut_{\mathcal{L}}(\mathfrak{D}/M^G)$, I suppose.
- 32_{11+12}: Typo: It should be twice τ instead of θ.
- Proof of Theorem 2.2.51: One may give a simpler (and also more enlightening) proof of this result, if one observes first that If P is a PAC substructure of a (large) stable structure $\mathfrak{D}$, then any elementary restriction $P' \preccurlyeq P$ is also a PAC substructure of $\mathfrak{D}$. Indeed, if $P' \subseteq M'$ is a regular extension, where M' is a small substructure of $\mathfrak{D}$, one may place M' such that

M' and P are independent over P'. Setting $M := dcl(M'P)$, it follows (by stationarity of $tp(m/P')$ for every finite tuple m from M') that $P \subseteq M$ is regular. Then $P \leq_1 M$, so $P' \leq_1 M$ and in particular $P' \leq_1 M'$.
Also, in your proof, the structure $\overline{P}$ constructed in line 2 of page 34 may not be saturated enough. (It woulld be enough to replace the underlying chain by one of length κ^+.)

- It would seem more adequate to prove Lemma 2.3.6 and then get Lemma 2.3.5 as a corollary.
- Remark 2.3.8: The formulation is not clear. You might write something like "If $A := acl^{\mathfrak{D}}_{\mathcal{L}}(\emptyset) \cap M_1 = acl^{\mathfrak{D}}_{\mathcal{L}}(\emptyset) \cap M_2$ and $\overline{\tau}_1$ and $\overline{\tau}_2$ agree on A, then..."
- 36_{10}: It seems that the argument needs Lemma 2.3.6, and not just Lemma 2.3.5. Also, it would seem more natural to state 2.3.10 and 2.3.11 in one lemma.
- 37_{4+5}: The sentence in parentheses is not clear, and also not needed. Moreover, you may want to add in 37_3: "...Lemma 2.2.19 for $N = acl^{\mathfrak{C}}_{\mathcal{L}^G}(A') \subseteq acl^{\mathfrak{D}}_{\mathcal{L}}(A')$ and the action of the group H, it follows..."
- The proof of (vi) on pp. 38+39 seems much too complicated and may certainly be simplified considerably (since for any E'-embedding of $(N, \overline{\rho})$ into $\mathfrak{C}$ one gets the independence result you claim by automorphism invariance of the independence relation in $\mathfrak{D}$).
- 42_4: Should it be "... of $acl^{\mathfrak{C}}_{\mathcal{L}^G}(Mac_1)$ and C" (instead of $acl^{\mathfrak{C}}_{\mathcal{L}^G}(Mac_1)$ or even $acl^{\mathfrak{C}}_{\mathcal{L}^G}(Mbc_1)$)?
- Example 2.3.25: It is not true that T^{mc}, the theory of an infinite (pure) set, eliminates imaginaries. It only eliminates them weakly.
- Line 5 of proof of 2.3.27: It might be clearer to write "...let D_1' be the $\mathcal{L}$-definable closure..."
- It would be good to give the argument for Corollary 2.3.38, namely that G finite implies that M is Galois bounded (since it is a finite extension of M^G which is Galois bounded), so bounded, so $\mathfrak{C}$ is bounded.
- In Corollary 2.3.3 (given the previous remark), it might be clearer to state in part (1): "If $\mathfrak{C}$ is bounded, in particular if G is finite, ..."
- In Remark 2.3.32 (and also in line 2 of Fact 2.3.33 and in line 2 of Corollary 2.3.34), it should be "...let q be an extension..." instead of "...let q be its extension..."
- In Theorem 2.3.35, it might be useful if you remarked that the hypothesis of the theorem implies in particular that T^{mc}_G is simple.
- 47^1: It would be even clearer if you wrote "... and so Gd..." instead of "...and so d...".
- 47_{11}: Typo: It should be "final".
- Lemma 3.1.8: You mean "G-transformal field" here.
- Remark 3.3.3: The last equivalence is wrong as stated: It should be "$trdeg(Ga(\langle GB, GC\rangle) = trdeg(Ga/\langle GC\rangle)$", i.e. $\langle GC\rangle$ instead of $\langle GB\rangle$ at the end (which is certainly a typo), but more importantly Ga instead of a. Indeed, consider the $\mathbb{Z}/2$-action on $\mathbb{Q}(X,Y)$ which interchanges X and Y. Letting $C = \mathbb{Q}$, $B = X+Y$ and $a = X$, one gets $trdeg(a(\langle GB, GC\rangle) = 1 = trdeg(a/\langle GC\rangle)$, but $trdeg(Ga/\langle GB, GC\rangle) = 1 < 2 = trdeg(Ga/\langle GC\rangle)$.
- First line of the proof of Theorem 3.3.7: What you need is that the interpretation is quantifier-free both ways (which is of course the case).
- In the first commuting diagram on page 65, the vertical on the right should probably be $\mathbb{D}[[Y]] : R[[Y]] \to R[[X, Y]]$.
- In the second (and third) commuting diagram on page 65, elements v_m and w_m appear which have not yet been introduced. (Only on page 66 are they introduced.) Also, on top of page 66, $Fr^m_{\mathbb{G}_a}$ is used without having been introduced.
- In the commutative diagram on top of page 66, it should be X in place of R.